Optimal Control of Information Epidemics

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***Abstract*—In this work we address two problems concerning information propagation in a population: a) how to maximize the spread of a given message in the population within the stipulated time and b) how to create a given level of buzz- measured by the fraction of the population engaged in conversation on a topic of interest- at a specified time horizon. Arising in the context of two rather disparate networks- social and wireless vehicular networks, their importance in campaigning on social networks and security in vehicular networks can only be understated. Taking a mean-field route, we pose the two problems as continuous-time deterministic optimal control problems. We characterize optimal controls, present some numerical results and provide practical insights. Interestingly, the problems we address are antithesis to those in disease epidemics, which need to be contained rather than actively spread.**

I. Introduction

The advent of Web 2.0 technologies and an explosion in social media applications in recent years have revolutionized social networks. While these networks have existed since time immemorial, technological advances have made it possible for people to connect, communicate and share with each other easily, (almost) instantaneously, using a variety of media, from anywhere, anytime and without geographical or cultural boundaries. Social networks have also expanded in scope to include brands, businesses and news media in peoples' on-line social circles. It is, therefore, not surprising that campaigners- be them social, political or marketing- find (on-line) social networks attractive forums to reach out to masses[1]. Campaigners, typically, have two main concerns among others[2]: a) to be able to communicate a piece of information, say, a marketing message, to as large a fraction of the population as possible, in stipulated time, and b) to be able to engage people in conversations on a topic (which entails exchange of multiple messages on the same subject but not necessarily the same content) so that by a given time the impact of promotional activities ("buzz") reaches a desired level. Neither task is easy however, since with the huge amount of information generated today, it is difficult to get eyeballs on specific messages and retain peoples' interest.

Similar concerns arise in vehicular networks (or more generally, wireless sensor networks) as well. Wireless vehicle-to-vehicle (V2V) communications aim to improve driver safety, by having vehicles exchange messages pertaining to kinematic parameters, road condition and traffic information [1]. Alerts arising out of, say, a hazardous road condition need to be conveyed to vehicles approaching the target area by certain time to be of any use to the drivers. Vehicular networks also need to support dissemination of system-wide alerts/updates from a security infrastructure (such as based on public key cryptography (PKI)), which ensures that the messages acted upon by the safety critical V2V applications are protected from spoofing, alteration, and replay. One-time updates (e.g., fixes, queries) are required to reach a large fraction of vehicles in a given time whereas recurrent updates (e.g., certificate revocation lists [2]) need to be disseminated in such way that system's vulnerability to attacks is as low as possible. Here again the two tasks are not easy given the expected volume of messages/alerts, limited infrastructure and bandwidth for transmissions and idiosyncrasies of the wireless channel.

In this paper, we address the two problems described above in a rather general setting. We consider a population of "smart" agents (i.e., people, vehicles or sensors) with an underlying network specifying conduits for information (or message) transfer. Each agent is a potential producer, consumer and relay of messages. Message propagation is epidemic in nature, i.e., if an agent communicates a message to its neighbors, they *may* in turn communicate it to their neighbors and so on. Agents can also be communicated messages directly and recruited for spreading them, albeit at a cost. In the case of a social networks this cost may be seen as the cost of incentives offered to people to forward messages to their friends whereas in a vehicular network it could be the cost of using fixed infrastructure (e.g., road side units) to communicate with the vehicles. A budgetary constraint implies that not all agents can be given message(s) directly. This necessitates an optimal control of recruitment and, thereby, epidemic transmission processes in the population. We pose the two problems described earlier as follows.

- P1 **Message dissemination**: Given a budget constraint, maximize the fraction of the population receiving a tagged message by the stipulated time, by controlling the number of recruits over time.
- P2 **Topic diffusion**: Minimize the cost incurred to create a given level of buzz, i.e., the fraction of population engaged in conversion on a topic, at a specified time, by controlling the number of recruits over time.

To make the problems mathematically precise, we take

[1]See numerous blogs devoted to discussing these issues, most notably Mashable, Socialsignal and Internet marketing blog.

[2]To be able to contain rumors/negative news is one such.

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the mean-field approach and obtain a system of ordinary differential equations (ODEs) to describe the system evolution. Given that the networks under consideration are large, we believe that the thermodynamic limit (i.e., taking the size of the population to infinity) to obtain the system of ODEs is justified and the controls we derive have practical significance. In order to simplify our analyses and to deliver key insights, in this paper, we will underplay the structure of the networks and characterize them only through the average degree[3]. We are, therefore, led to the SIR epidemic system for problem P1 and the SIS epidemic system for problem P2. The two problems are then formulated as continuous-time deterministic optimal control problems. We characterize optimal controls using Pontryagin Maximum principle, validate them through numerical studies and provide some practical insights.

We show that in the case of problem P1 (message dissemination), the optimal control has switching behavior- it is optimal to recruit for some initial duration till the budget is spent and then stop. In problem P2 (topic diffusion), the optimal strategy shows different behaviors, depending on whether the population can sustain interest in the topic (this is formalized in terms of the stability of a certain equilibrium). If the topic cannot be sustained, then it is optimal to start recruiting just before the desired time horizon. On the other hand, if the topic can be sustained, then the optimal strategy is to recruit for some initial duration and then stop.

This paper is organized as follows. Related work is reviewed in Section II. In Section III and Section IV we discuss the message dissemination problem and the topic diffusion problem respectively. Numerical results are provided in Section V. We conclude in Section VI.

II. Related Work

Our work is closely related to epidemic processes for which there is a long history of work [4]. Not surprisingly, there is also a significant body of work on the control of epidemics given its practical importance; see, for example, [5], [6] for some optimal control formulations. Epidemic models and control formulations have also been investigated in the context of *on-line* viruses and worms (see [7] and reference therein). Though the models we analyze in this paper are essentially the epidemic models, the problems we address are antithesis to those in the line work of described above- disease/virus needs to be contained whereas in our case information is to be widely spread. Problems concerning maximization of spread of rumors and malware have attracted attention recently: impulse control of rumor spreading is studied in [8] and optimal control of malware propagation assuming a variation of the SIR model is investigated in [9]. Our work differs from the former both in the model and the problem formulation (it assumes Daley-Kendall and Maki-Thomson models) whereas it differs from the latter in terms of the control used (it considers 'kill rate' and 'infection rate' as controls). Moreover, we also investigate the control formulation for the SIS system.

In the case of social networks, research has focused more on understanding information diffusion [10] whereas in the case of wireless mobile networks the focus is more on design and evaluation of algorithms/protocols for dissemination (e.g., [11])) and distributed computation (see [12] for gossip algorithms). We believe that our problem formulations and control strategies are first of their kind in the context of social and vehicular networks.

III. Message Dissemination

Consider a population of N agents. The interconnections among them are specified by graph $\mathcal{G}$, which is drawn randomly from the set of undirected graphs of size N and average degree k. The agents become active at random times to transmit messages. "Activity" process of each agent is modeled as a Poisson process of rate 1 and is assumed to be independent of those of other agents. We use epidemiological terminology and refer to an agent as *susceptible* if she has not received the tagged message. An agent becomes *infected* when she receives the tagged message and actively spreads it. An infected agent is said to have *recovered* if she stops spreading the tagged message. If the message is communicated by one of her social contacts, a susceptible agent turns into an infected one with probability β'. In vehicular networks, β' is simply the probability of successful reception whereas in social networks, it can be seen as the probability of conversion due under social influence [13]. A susceptible agent may also become infected by direct "recruitment"- when a susceptible agent becomes active, with probability u, she is sent the tagged message directly and recruited to act as a spreader at a given cost. An infected agent recovers with probability $v > 0$ upon being active.

Now given a fixed budget $B > 0$ to recruit agents, the objective is to exercise control u so that starting with a susceptible population, the number of recipients of the tagged message by given time $T < \infty$ is maximized. In view of the large scale of the networks under consideration, we take the following approach. Let S (resp. I) denote the number of susceptible (resp. infected) agents in the population. Assuming that agents are indistinguishable except for their states, it follows that (S, I) is a continuous-time Markov chain. Taking a mean-field limit [14], we obtain the following (SIR epidemic) system of ODEs, where s (resp. i) denotes the fraction of susceptible (resp. infected) population, $\beta := \beta' k$ and $u(\cdot)$ is the recruitment control[4].

$$\dot{s}(t) = -\beta s(t)i(t) - u(t)s(t) \quad (1)$$
$$\dot{i}(t) = \beta s(t)i(t) - vi(t) + u(t)s(t) \quad (2)$$
$$\dot{r}(t) = vi(t)$$

[3] In the context of vehicular networks it may be more reasonable to assume homogeneous mixing directly than imposing a network structure (see [3] and references therein).

[4] While a mean field limit can be established for the *controlled* Markov chain on the lines of [15], we do not do so here. We take the mean field limit of the stochastic system and treat parameter u as control, for controlling the above system of ODEs.

Observe that $\dot{s}+\dot{i}+\dot{r}=0$. Therefore, it suffices to consider only (1) and (2). Let $\Omega := \{(s,i)|0 \le s,i \le 1,\ s+i \le 1\}$ and $\mathcal{U} := \{u|0 \le u \le \bar{u}\}$. Henceforth we scale $\bar{u}$ to 1 without loss of generality.

The cost of recruitment is linearly proportional to the recruitment effort u. With appropriate normalization the budget constraint can be expressed as $\int_0^T u(t)dt \le B$. It is assumed that $B < T$, otherwise the problem has a trivial solution that $u(t) = 1,\ \forall t$. This isoperimetric constraint can be converted into a (more amenable) constraint on the terminal state by defining a (dummy) state variable $w(t)$ with $\dot{w}(t) = -u(t)$ and $w(0) = B$. The budget constraint is then simply stated as $w(T) \ge 0$.

Let $\mathbf{x}^T(t) := (s(t), i(t), w(t))$ denote the state vector and

$$f(\mathbf{x}(t), u(t)) := \begin{bmatrix} -\beta s(t)i(t) - u(t)s(t) \\ \beta s(t)i(t) - vi(t) + u(t)s(t) \\ -u(t) \end{bmatrix} \quad (3)$$

The problem can now be formally stated as

$$\text{Maximize } 1 - s(T) \quad (4)$$

subject to $\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), u(t))$ and the following constraints on state and control variables: for all $0 \le t \le T$, $\mathbf{x}(t) \in \Omega \times R_+,\ u(t) \in \mathcal{U}$, $\mathbf{x}(0) = (1, 0, B)$ and $w(T) \ge 0$.

We say that $(\mathbf{x}^*(\cdot), u^*(\cdot))$ is an *optimal pair of state and control trajectory* if $u^*(\cdot)$ is piecewise continuous, $\mathbf{x}^*(\cdot)$ is continuous and piecewise continuously differentiable and $u^*(\cdot)$ maximizes the objective (4) under the stated constraints. Our main result is the following.

Proposition 3.1: $u^*(t) = 1$ for $0 \le t \le B$ and $u^*(t) = 0$ for $B < t \le T$.

In the following we present a proof of Proposition 3.1. We start by observing from (3) that a unique solution exists in $[0, T]$ for any initial condition $\mathbf{x}(0) \in \Omega \times R_+$. Moreover, Ω is positively invariant: $\{(s, 0)|0 \le s \le 1\} \in \Omega$ is the set of equilibria; $\dot{s} = 0$ whenever $s = 0$ and on the boundary $s + i = 1$, $\dot{s} + \dot{i} \le 0$. Therefore, a solution starting from any initial condition $(s, i) \in \Omega$(and in particular $(1, 0)$) remains confined to Ω. Hence, the path-wise state constraints can be ignored from the problem statement.

Note that the formulation involves only terminal cost and the system (3) is affine in control. Existence of an optimal control is then established by Filippov-Cesari theorem [16] which guarantees that the reachable set is compact and by noting that the cost is continuous function of the state. We derive the structural form of an optimal control as follows.

Denoting by $\mathbf{p}^T(t) = (p_1(t), p_2(t), p_3(t))$ the vector of adjoint variables, from (3) and (4) the Hamiltonian for our problem can be written as

$$\begin{aligned} H(\mathbf{x}(t), \mathbf{p}(t), u(t)) &= \beta(p_2(t) - p_1(t))s(t)i(t) - p_2(t)vi(t) \\ &\quad +((p_2(t) - p_1(t))s(t) - p_3(t))u(t) \end{aligned} \quad (5)$$

The Maximum principle [16] provides the following necessary conditions for $(\mathbf{x}^*(\cdot), u^*(\cdot))$.

Adjoint condition: There exists continuous and piecewise continuously differentiable function $\mathbf{p}^*(\cdot)$ that solves the following system of ODEs at all t where $u^*(\cdot)$ is continuous.

$$\dot{p}_1(t) = (p_1(t) - p_2(t))(\beta i(t) + u(t)) \quad (6)$$

$$\dot{p}_2(t) = \beta(p_1(t) - p_2(t))s(t) + p_2(t)v \quad (7)$$

$$\dot{p}_3(t) = 0 \quad (8)$$

Transversality condition:

$$p_1^*(T) = -1,\ \ p_2^*(T) = 0,\ \ p_3^*(T) \ge 0,\ \ p_3^*(T)w^*(T) = 0 \quad (9)$$

Hamiltonian maximizing condition: For all $t \in [0, T]$ and $u \in \mathcal{U}$

$$H(\mathbf{x}^*(t), \mathbf{p}^*(t), u^*(t)) \ge H(\mathbf{x}^*(t), \mathbf{p}^*(t), u) \quad (10)$$

From (5) and (10) we get

$$u^*(t) = \begin{cases} 0 & (p_2^*(t) - p_1^*(t))s^*(t) < p_3^*(t) \\ 1 & (p_2^*(t) - p_1^*(t))s^*(t) > p_3^*(t) \end{cases}$$

$u^*(t)$ takes an arbitrary value in $\mathcal{U}$ in case $p_2^*(t) - p_1^*(t))s^*(t) = p_3^*(t)$. (8) implies that $p_3(t)$ is constant. Denote it by c and denote by $\phi(t)$ the "switching function" $(p_2^*(t) - p_1^*(t))s^*(t) - c$.

Let H_t^* denote the Hamiltonian at time t along $(\mathbf{x}^*(\cdot), u^*(\cdot))$. Then from (9) we get $H_T^* = \beta s^*(T)i^*(T) + u^*(T)(s^*(T) - c)$. Now $T < \infty,\ B > 0$ and $(s(0), i(0)) = (1, 0)$. Therefore, $s^*(T) > 0,\ i^*(T) > 0$ and hence $H_T^* > 0$. Since the Hamiltonian is constant for autonomous systems, $H_0^* > 0$. Hence $\phi(0) > 0$, and so $u^*(0) = 1$.

Let τ be such that $\phi(\tau) = 0$. Then from (5) we have

$$\begin{aligned} H_\tau^* &= \beta(p_2^*(\tau) - p_1^*(\tau))s^*(\tau)i^*(\tau) - p_2^*(\tau)vi^*(\tau) \\ &= c\beta i^*(\tau) - p_2^*(\tau)vi^*(\tau) \end{aligned}$$

$H_\tau^* > 0$ implies that $p_2^*(\tau) < \frac{c\beta}{v}$. This, in turn, implies that $\dot{p}_2^*(\tau) = (-c\beta + p_2^*(\tau)v) < 0$ (see (7)). Note that $\phi(\cdot)$ is continuous and piecewise continuously differentiable. From (3), (6) and (7) we get

$$\dot{\phi}(\tau) = s^*(\tau)\dot{p}_2^*(\tau) < 0$$

This implies that $\phi(\tau) = 0$ for at most one value of τ since at every "crossing" the slope is negative (there is no singular interval). Let t^* denote this value. $t^* < T$ since $B < T$. For $T \ge t > t^*$, $\phi(t) < 0$; hence $u^*(t) = 0$. As $\phi(0) > 0$, $t^* > 0$ and continuity implies that for $0 \le t \le t^*$, $\phi(t) \ge 0$ and hence $u^*(t) = 1$. $u^*(\cdot)$ is, thus, a bang-bang control. Then clearly $t^* = B$ otherwise the budget is underutilized and the solution is not optimal. The proposition is, thus, established.

IV. Topic Diffusion

The underlying stochastic model in this case is similar to the one described in Section III. The difference is that an infected agent *does not recover* but goes back to being susceptible. In the case of social networks, the interpretation is that during the infected phase, an agent participates in the conversation (by transmitting messages on the topic). She becomes susceptible when she withdraws from the discussion. She can again be pulled into the discussion, thereby, again becoming infected. In vehicular networks, being in infected state may mean that a (pre-designated relay) vehicle has the latest security update (say, certificate revocation list). Over a period of time, it becomes outdated and the vehicle is rendered susceptible to attacks. The reception of the latest update makes it infected again[5]. As in Section III, the agents can be infected by recruiting them at a cost; when a susceptible agent becomes active, with probability u, she is recruited to be a participant.

In the following, we use the social network terminology. The *buzz* at time t is measured in terms of the fraction of the population engaged in discussing the topic, i.e., infected. The objective is to minimize the cost of creating a certain level of buzz at given time T, the cost being that of recruiting agents. Taking the same approach as in Section III, we obtain the following (SIS epidemic) system of ODEs to be controlled through recruitment effort $u(\cdot)$.

$$\begin{aligned} \dot{s}(t) &= -\beta s(t)i(t) + vi(t) - u(t)s(t) \\ \dot{i}(t) &= \beta s(t)i(t) - vi(t) + u(t)s(t) \end{aligned} \qquad (11)$$

Since $s(t) + i(t) = 1, \forall t$, it suffices to consider only (11). Replacing $s(t)$ with $1 - i(t)$, we obtain

$$\dot{i}(t) = -\beta i(t)^2 + (\beta - v - u(t))i(t) + u(t) \qquad (12)$$

As in Section III the cost of recruitment is taken linearly proportional to the recruitment effort u and is normalized to 1. Denoting by $b > 0$ the target level of buzz, the problem is now stated formally as follows.

$$\text{Maximize } -\int_0^T u(t)dt \qquad (13)$$

subject to (12) and the following constraints on state and control variables: for all $0 \le t \le T$, $i(t) \in \Omega := \{i | 0 \le i \le 1\}$, $u(t) \in \mathcal{U} := \{u | 0 \le u \le 1\}$, $i(0) = 0$ and $i(T) \ge b$.

Let y_1 and y_2 denote the roots of $i^2 + \frac{1+v-\beta}{\beta} i - \frac{1}{\beta} = 0$.

Proposition 4.1: Buzz level b is achievable at T if and only if $T \ge \frac{1}{\beta(y_2 - y_1)}[\ln(1 - \frac{b}{y_1}) - \ln(1 - \frac{b}{y_2})]$.

Proposition 4.1 simply states that any buzz level not greater than that reachable by applying $u(t) = 1, \ \forall t$ is achievable at T. We assume that b is achievable at T and that strict inequality holds in Proposition 4.1 (else the solution is trivial).

Let $(i^*(\cdot), u^*(\cdot))$ denote an optimal state and control pair (optimality as defined in Section III).

Proposition 4.2:
1) If $\frac{v}{\beta} > 1$, then there exists $0 < t^* < T$ such that $u^*(t) = 0$ for $0 \le t \le t^*$ and $u^*(t) = 1$ for $t^* < t \le T$.
2) If $\frac{v}{\beta} < 1$, then there exist $0 < t^* < \tilde{t} \le T$ such that $u^*(t) = 0$ for $t^* < t \le \tilde{t}$ and $u^*(t) = 1$ otherwise. If $b \le (1 - \sqrt{\frac{v}{\beta}})$, $\tilde{t} = T$. If $b \ge (1 - \frac{v}{\beta})$, $\tilde{t} < T$.

In the rest of the section, we prove Proposition 4.2. First, observe that $\dot{i} \ge 0$ for $i = 0$ and $\dot{i} < 0$ for $i = 1$. Therefore, Ω is positively invariant and the path-wise state constraint can be ignored. Second, the terminal state constraint must be met with equality, i.e., $i^*(T) = b$, for optimality.

The Hamiltonian for this problem is

$$\begin{aligned} H(i(t), p(t), p_0, u(t)) &= p(t)(-\beta i(t)^2 + (\beta - v)i(t)) \\ &\quad + (p(t)(1 - i(t)) - p_0)u(t) \end{aligned} \qquad (14)$$

$p(t)$ denotes the adjoint variable and p_0 is the multiplier. From the Maximum principle [16] we obtain the following necessary conditions for $(i^*(\cdot), u^*(\cdot))$.

Adjoint condition: There exist $p_0^*, p^*(t) \ge 0$, $(p_0^*, p^*(t)) \ne \mathbf{0}$. $p^*(t)$ is a continuous and piecewise continuously differentiable function that solves the following ODE at all t where $u^*(\cdot)$ is continuous.

$$\dot{p}^*(t) = p(t)(2\beta i(t) + v - \beta + u(t)) \qquad (15)$$

Transversality condition:

$$p^*(T) \ge 0, \ \ p^*(T)(i^*(T) - b) = 0$$

Hamiltonian maximizing condition: For all $t \in [0, T]$ and $u \in \mathcal{U}$

$$H(i^*(t), p^*(t), p_0^*, u^*(t)) \ge H(i^*(t), p^*(t), p_0^*, u)$$

This yields the following form for the optimal control.

$$u^*(t) = \begin{cases} 0 & p^*(t)(1 - i^*(t)) < p_0^* \\ 1 & p^*(t)(1 - i^*(t)) > p_0^* \end{cases}$$

$u^*(t)$ takes an arbitrary value in $\mathcal{U}$ in case $p^*(t)(1 - i^*(t)) = p_0^*$.

If $p_0^* = 0$ then $p^*(t) > 0, \forall t$ which implies that $u^*(t) = 1, \forall t$. This cannot be optimal given our assumption that b is achievable with strict inequality in Proposition 4.1. Hence the problem is *normal* and we set $p_0^* = 1$. If $p^*(0) = 0$ then $p^*(t) = 0, \forall t$ (see (15)). This would imply that for $0 \le t \le T$, $u^*(t) = 0$, which is clearly not optimal. Therefore, $p^*(0) > 0$. Let $\psi(t) := p^*(t)(1 - i^*(t)) - 1$. From (12) and (15) we get

$$\dot{\psi}(t) = p^*(t)(v - (i^*(t) - 1)^2\beta) \qquad (16)$$

Suppose $\frac{v}{\beta} > 1$. Then for all $0 \le t \le T$, $\dot{\psi}(t) > 0$. If $p^*(0) > 1$ then this implies that $u^*(t) = 1, \forall t$, which is not optimal in view of our assumption regarding strict reachability of b. Hence $p^*(0) < 1$ and there exists $0 < t^* < T$ such that $\psi(t) > 0$ for $t^* < t \le T$ (otherwise $u^*(t) = 0, \forall t$, which is clearly not optimal). Therefore, $u^*(t) = 0$ for $0 \le t \le t^*$ and $u^*(t) = 1$ for $t^* < t \le T$.

Now consider the case that $\frac{v}{\beta} < 1$. Let $\bar{i} := (1 - \frac{v}{\beta})$. $\bar{i} < 1$. As in Section III, we denote by H_t^* the Hamiltonian at time t along the optimal state and control trajectory. Writing (14) in a more suggestive form, we have $H_t^* = \psi(t)u^*(t) - \beta p^*(t)i^*(t)(i^*(t) - \bar{i})$. Recall that $i^*(0) = 0$, $i^*(T) = b$ and note that $i^*(t) \uparrow b$. Therefore, there must exist $0 < \hat{t} < T$ such that $p^*(\hat{t}) > 0$ and $0 < i^*(\hat{t}) < \bar{i}$. Then $H_{\hat{t}}^* > 0$. Since the Hamiltonian is constant for autonomous systems, $H_t^* > 0, \forall t$. Therefore, $p^*(0) > 1$ (otherwise $H_0^* = 0$) and $u^*(0) = 1$.

[5]Thus, in our formulation, being infected is actually good for health!

Observe from (16) that $\dot{\psi}(t) < 0$ if $i^*(t) < (1 - \sqrt{\frac{v}{\beta}})$ and $\dot{\psi}(t) > 0$ if $i^*(t) > (1 - \sqrt{\frac{v}{\beta}})$. Let τ be such that $\psi(\tau) = 0$. Then $H^*_\tau > 0$ implies that $i^*(\tau) < \bar{i}$. Since $i^*(\cdot)$ is monotonic, the above implies that there can be at most two "crossings" of $\psi(\cdot)$. Let t^* and $\tilde{t}$ denote resp. the first time and the second time $\psi(\cdot) = 0$. Since $\psi(0) > 0$, continuity of $\psi(\cdot)$ implies that $u^*(t) = 1$ for $0 \le t \le t^*$. For $t^* < t \le \tilde{t}$, $\psi(t) < 0$ hence $u^*(t) = 0$. For $\tilde{t} < t \le T$, $\psi(t) > 0$ implying that $u^*(t) = 1$.

Suppose $b \ge \bar{i}$. Then $H^*_T > 0$ implies that $u^*(T) = 1$. Since $i^*(t) \uparrow b$ and $\dot{\psi}(t) > 0$ for $i^*(t) > (1 - \sqrt{\frac{v}{\beta}})$, $\tilde{t} < T$. When $b \le (1 - \sqrt{\frac{v}{\beta}})$, $\psi(t) \le 0$ for $t^* < t \le T$ since the condition means that $\dot{\psi}(t) < 0, \forall t$ (see (16)).

Now in this problem, a relatively simple form of (12) allows us to derive value functions explicitly and, thereby, verify the Hamilton-Jacobi-Bellman (HJB) equations. Since the algebra is cumbersome, here we discuss only the case when $\frac{v}{\beta} > 1$. Suppose that the initial state is i_0 and consider the control

$$u^*(t) = \begin{cases} 0 & 0 \le t \le t^* \\ 1 & t^* < t \le T \end{cases}$$

t^* is given by the following nonlinear equation.

$$y_2 + \frac{y_1 - y_2}{1 - \frac{i(t^*) - y_1}{i(t^*) - y_2} e^{-\beta(y_1 - y_2)(T - t^*)}} = b, \quad (17)$$

where, $i(t^*) = \frac{\bar{i}}{1 - (1 - \frac{\bar{i}}{i_0}) e^{-\beta \bar{i} t^*}}$ and, as earlier, y_1 and y_2 denote the roots of $i^2 + \frac{1+v-\beta}{\beta} i - \frac{1}{\beta} = 0$. Let i be a state on the optimal trajectory from i_0 and let t be the corresponding time such that $0 < t < t^*$. Then $u^*(t) = 0$ and the value function $V(i,t) = t^* - T$. $i(t^*)$ can be obtained in terms of (i,t) as $i(t^*) = \frac{\bar{i}}{1 - (1 - \frac{\bar{i}}{i}) e^{-\beta \bar{i}(t^* - t)}}$. From (17) we obtain (via the Implicit function theorem) $\frac{\partial V(i,t)}{\partial i} = -g(i,t,t^*) \frac{\bar{i}}{i^2}$ and $\frac{\partial V(i,t)}{\partial t} = -g(i,t,t^*) \beta \bar{i} (1 - \frac{\bar{i}}{i})$ for some function $g(i,t,t^*)$. It is easy to see that

$$\frac{\partial V(i,t)}{\partial t} + (-\beta i^2 + (\beta - v) i) \frac{\partial V(i,t)}{\partial i} = 0$$

Thus, the HJB equation is indeed satisfied. Now if $t^* < t \le T$, then $u^*(t) = 1$ and the HJB equation for $V(i,t) = t - T$ is trivially verified.

V. Numerical Results

In this section, we present numerical studies which validate our theoretical results and provide some insights into them. The numerical results are obtained by discretizing problems (4) and (13), posing them as nonlinear constrained optimization problems and employing a gradient descent algorithm to arrive at an optimal solution. Time horizon T and discretization step-size are fixed arbitrarily to 10 and 0.1 respectively. While discretization errors are apparent (and are to be expected), our results seem to fit the theoretical analyses quite well.

In Section III, we showed that the optimal control for message dissemination is bang-bang, and the optimal strategy

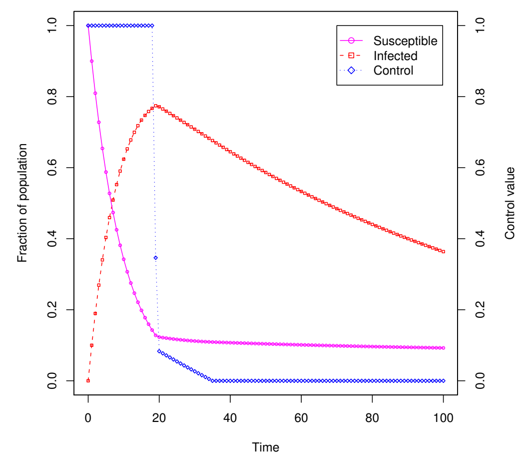

Fig. 1. Variation of the infected ($i(t)$) and the susceptible ($s(t)$) population with time. The optimal strategy is to recruit initially till the budget is spent and then stop. $B = 2$, $v = 0.1$, $\beta = 0.05$. By the end of time $T = 10$ (discretized into 100 steps), about 9% are still susceptible.

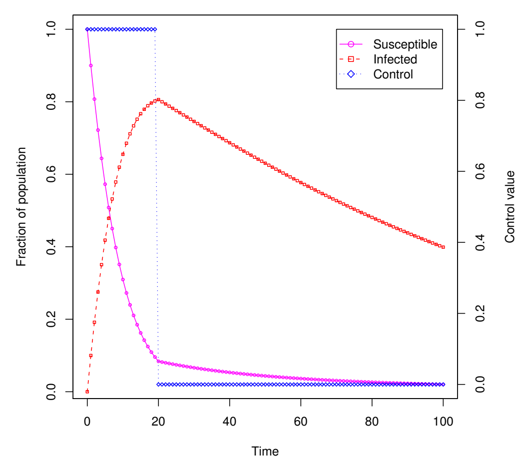

Fig. 2. Variation of the infected ($i(t)$) and the susceptible ($s(t)$) population with time. The optimal strategy is to recruit initially till the budget is spent and then stop. $B = 2$, $v = 0.1$, $\beta = 0.3$. By the end of time $T = 10$ (discretized into 100 steps) only 2% are susceptible.

is to recruit "at full throttle" till the budget is spent and then stop. The idea is get the population infected as fast as possible. This is affirmed in Figures 1 and 2. We fix $B = 2$, $v = 0.1$ and consider two values of β- 0.05 and 0.3. It is well known that an SIR epidemic occurs if $\frac{v}{\beta} < 1$ [4]. Thus, even if the budget is the same, when $\beta = 0.3$, an epidemic occurs in the population and the fraction of population that has not received the message till time T, i.e., $s(T)$, is only 2% (Figure 2) whereas when $\beta = 0.05$, there is no "epidemic gain" and $s(T)$ is 9% (Figure 1).

For diffusion of a topic, the optimal control is also bang-bang. However, switching exhibited by it seems to depend on $\frac{v}{\beta}$ and the target buzz b. For an SIS epidemic system, it is known that the disease persists if $\frac{v}{\beta} < 1$ and dies down if $\frac{v}{\beta} > 1$. Mathematically, this means that in the former case $(s,i) = (1,0)$ is globally asymptotically stable equilibrium whereas in the latter case it is $(s,i) = (\frac{v}{\beta}, 1 - \frac{v}{\beta})$. Thus, when

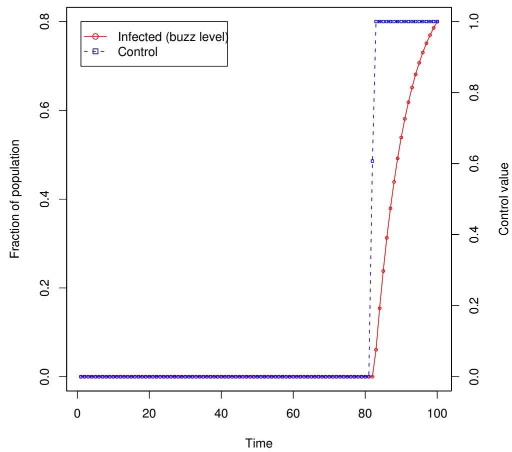

Fig. 3. Variation of the infected population with time ($i(t)$). $v = 0.1,\ \beta = 0.05$. The optimal strategy is to start recruiting just before the desired time horizon.At $T = 10$ (discretized into 100 steps), the target buzz level of 80% is reached.

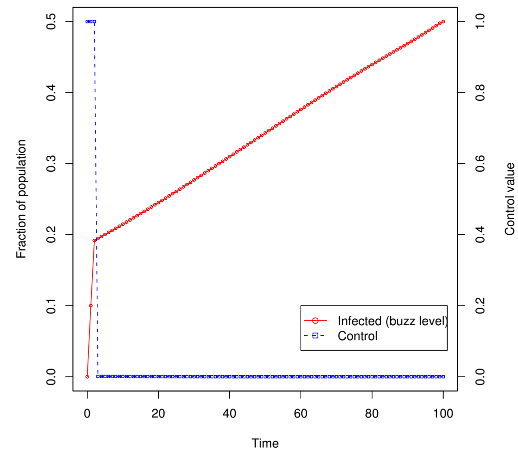

Fig. 4. Variation of the infected population with time ($i(t)$). $v = 0.1,\ \beta = 0.3$. The target buzz level is $50\%(> (1 - \sqrt{\frac{v}{\beta}}))$. The optimal strategy is to recruit for some initial duration and then stop. Time horizon is $T = 10$ (discretized into 100 steps).

$\frac{v}{\beta} > 1$, to create buzz b by T, it is optimal to start just ahead of time otherwise it will fizzle out. This is demonstrated in Figure 3. We set $v = 0.1,\ \beta = 0.05$ and $b = 0.8$.

On the other hand, if $\frac{v}{\beta} < 1$, infecting the population initially for some duration (t^*) ensures that the epidemic is underway, and it is possible to reach the target buzz b "on its way" to $(1 - \frac{v}{\beta})$ (this behavior of the optimal control may be construed as exhibiting a *tipping point*). This is indeed the case when $b \leq (1 - \sqrt{\frac{v}{\beta}})$. If $b > (1 - \frac{v}{\beta})$, then an extra "push" is required towards the end otherwise the buzz will end up near the stable point of $(1 - \frac{v}{\beta})$. For numerical studies we set $v = 0.1,\ \beta = 0.3$ and consider two targets- 0.5 and 0.8. These parameters ensure that the targets are achievable in $T = 10$ (maximum reachable target is 0.927). Shown in Figure 4 is the case for $b = 0.5$. Note that $(1-\sqrt{\frac{v}{\beta}}) < b < (1-\frac{v}{\beta})$. However, only initial recruitment suffices in this case. In Figure 5, $b = 0.8$ is considered ($b > (1 - \frac{v}{\beta})$) and the last push is apparent.

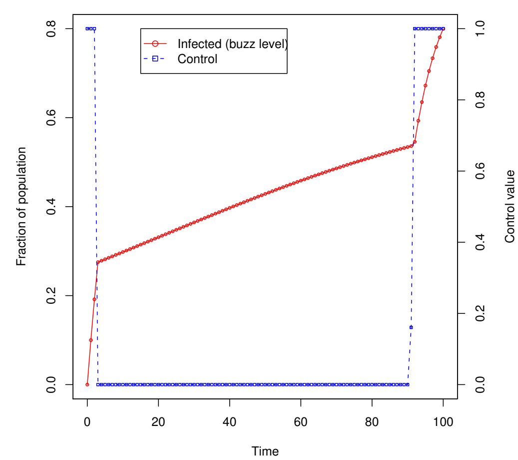

Fig. 5. Variation of the infected population with time ($i(t)$). $v = 0.1,\ \beta = 0.3$. The target buzz level is $80\%(> (1 - \frac{v}{\beta}))$. The optimal strategy is to recruit for some initial duration, stop and then recruit again till the end. Time horizon is $T = 10$ (discretized into 100 steps).

VI. Conclusion

In this paper we addressed two problems concerning information propagation in a population. We formulated them as continuous-time deterministic optimal control problems and derived the optimal controls. It turns out that in the case of message dissemination, the optimal strategy is to recruit for some initial duration till the budget is spent and then stop. For creating buzz (diffusion of topic), the optimal strategy shows different behaviors depending on whether the population can sustain interest in the topic. The numerical results seem to validate our theoretical analyses. This work can be extended in two directions: explicitly modeling the network structure to account for heterogeneities among agents and controlling other parameters such as "retention rate" (v) in addition to recruitment.

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