Stability of Transportation Networks Under Adaptive Routing Policies

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Abstract

Growing concerns regarding urban congestion, and the recent explosion of mobile devices able to provide real-time information to traffic users have motivated increasing reliance on real-time route guidance for the online management of traffic networks. However, while the theory of traffic equilibria is very well-known, fewer results exist on the stability of such equilibria, especially in the context of adaptive routing policy. In this work, we consider the problem of characterizing the stability properties of traffic equilibria in the context of online adaptive route choice induced by GPS-based decision making. We first extend the recent framework of “Markovian Traffic Equilibria” (MTE), in which users update their route choice at each intersection of the road network based on traffic conditions, to the case of non-equilibrium conditions, while preserving consistency with known existence and uniqueness results on MTE. We then exhibit sufficient conditions on the network topology and the latency functions for those MTEs to be stable in the sense of Lyapunov for a single destination problem. For various more restricted classes of network topologies motivated by the observed properties of travel patterns in the Singapore network, under certain assumptions we prove local exponential stability of the MTE, and derive analytical results on the sensitivity of the characteristic time of convergence to network and traffic parameters. The results proposed in this work are illustrated and validated on synthetic toy problems as well as on the Singapore road network with real demand and traffic data.

1 Introduction

The analysis of traffic network optima and equilibria under specific route choice models has historically motivated sustained research in the transportation community. The pioneering work of [28] has been foundational in that regard, and has been extended in a number of theoretical, numerical and experimental studies considering specific network topologies, path choice models, information model, rationality assumptions, optimality conditions (see for instance [2], [11], [6]).

Efficient mathematical programming models for solving traffic equilibria have also been proposed, starting from the seminal work of [5], who formulated the computation of the user equilibrium as a convex optimization problem. An associated variational formulation was later introduced in [11]. A number of computational and numerical frameworks have emerged from the dynamic traffic assignment community for the analysis of traffic phenomena, see [23] and references therein. For day-to-day dynamics, [16] present a method to compute the equilibrium in a Markov Chain framework. For a recent comprehensive description of theoretical results, algorithms and computational methods for the user equilibrium problem, we refer the reader to [20].

However traffic networks are not always at equilibrium. Dynamical models of non-equilibria conditions date back to the 1980’s with the pioneer work of [27], leveraging Lyapunov theory to analyze dynamical properties of a path choice model. In [21], the authors present detailed sufficient conditions for the stability of such path choice model (see also [7], [9]). Recent results from [18] exhibit conditions for asymptotic and exponential stability when the dynamics is modeled by a scalar conservation law. [19] present a comprehensive stability and controllability analysis for a similar model. For day-to-day dynamics, [17] provide results based on experimental simulations of drivers.

With the recent explosion in the number of GPS routing devices, recent work on shortest path and vehicle routing have been proposed for so-called adaptive routing models, in which the user may update his route choice at any node of the road network depending on traffic conditions. Real-time information is now common in traffic information systems, see [4] [12]. Algorithms to compute the optimal shortest path of a user using such real time information along the route are also of significant interest, see [22], [24], [25], [13], [14].

On the other hand, the question of the stability of traffic networks under adaptive route choice, in which users may update their decision at any node of the network, has received less attention. Concerns about the timescale of traffic variations under such adaptive routing policies motivated [10], which analyses a dynamical evolution of
traffic when users take into account real time information about the local congestion of roads only. In particular, the authors showed that an approximate Wardrop equilibrium is approached over time provided that the global preferences of path choice is slowly updated. In the framework of adaptive routing, traffic control is studied in [1] in which the authors prove that a certain category of algorithms (those minimizing the maximum volume) is stable.

In this article we study the stability of network equilibria under a realistic adaptive routing model, in which users consider global alternatives at each node of the network. We consider the framework of “Markovian Traffic Equilibria” (MTE) introduced in [3], which is consistent with the decision model developed in the routing community, see [24]. In a MTE framework, users leverage real time global information to choose the “best” path from each node of their current route. Imperfect information is also assumed, leading to a stochastic perception of link travel-times.

We focus our analysis on the case of exponential stability of MTE, i.e., we exhibit conditions under which a network subjected to a disturbance creating non-equilibria conditions, returns “very fast” to its equilibrium characterized by a MTE.

The main contributions of the work presented in this article are the following.

- We formulate a dynamical model of traffic flow in the framework of “Markovian Traffic Equilibria” in which users update their route choice at each node of the network depending on traffic conditions. The model proposed describes traffic dynamics outside of equilibrium.
- We propose a Lyapunov function candidate and prove that several of the technical conditions required for exponential stability can be obtained for any network type. We prove exponential stability of the MTE for a class of homogeneous tree networks. We then derive exponential stability results on parallel networks, and discuss how these results apply to the more limited case of a-priori route choice models.
- We present numerical results using synthetic data as well as real field data from Singapore. We illustrate that the convergence time predicted by the theory is observed numerically, and we show that the theoretical sensitivity to traffic parameters is also observed in numerical experiments. Numerical results on the relative convergence time of adaptive route choice models versus a-priori route choice models are also presented. Finally, numerical simulation of the time evolution of traffic conditions on the Singapore road network illustrate the potential applicability of our results to road traffic management.

The rest of this article is organized as follows. In Section 2 we define the notation and introduce the various models considered. In Section 3, we introduce a Lyapunov function candidate and present our Lyapunov stability analysis. Section 4 consists of technical results on a specific class of network topologies, while our main stability results are stated in Section 5. Finally, Section 6 provides numerical results using both synthetic and field data. Section 7 presents concluding remarks and comments.

2 Preliminaries

In this section we introduce the notation and present the preliminary results used in the rest of the article.

2.1 Static network model of equilibrium traffic

Let $G = (V, A)$ be a directed graph representing a traffic network. Let $V$ denote the set of $|V|$ nodes and $A$ the set of $|A|$ directed arcs. For all arcs $i \in A$ we denote $o(i)$ the origin node of arc $i$ and $d(i)$ its destination node. For all nodes $v \in V$ we use the following notation to refer to its incoming and departing arcs: $A^+_v = \{i \in A \mid o(i) = v\}$ and $A^-_v = \{i \in A \mid d(i) = v\}$. We also denote $i = A^+_o(i)$ and $i^+ = A^+_d(i)$ and $i^- = A^-_o(i)$.

![Network notation](image)

**Figure 1: Network notation.**

We assume that the network is as simple as possible while still capturing its essential aspects; in particular, if a node has only 1 incoming link and 1 outgoing, the node is pruned.

**Assumption 1.** Given a directed graph $G = (V, A)$ we assume for for each node $v \in V$, $|A^+_v| |A^-_v| \neq 1$. 


We consider a macroscopic model of traffic flow, and denote by \( f_a \) the flow on arc \( a \), expressed in vehicle per time unit. We denote by \( \tau_a \) and we call travel-time on arc \( a \) the time experienced by an element of flow (interpreted as a driver in an microscopic description) to drive through arc \( a \). The latency function \( s_a(\cdot) \) on arc \( a \) relates the travel-time to the current flow on arc \( a \),

\[
\tau_a = s_a(f_a),
\]

such that \( s_a(0) \) is the free-flow travel-time. We assume that the latency function is strictly increasing, continuous and differentiable. We consider a multi source - single destination model. We denote by \( g_v \) the exogenous traffic demand at node \( v \) with (unique) destination the sink node \( t \). We further denote \( |x| = \max_{1 \leq i \leq n} |x_i| \) for all \( x \in \mathbb{R}^n \), and \( B_h(\tau) \) the ball with center \( \tau \) and radius \( h \in \mathbb{R}_+ \).

### 2.2 Static model of route choice

We model route choice decisions using *split factors*, defined at decision points, and characterizing the allocation of elements of flow across possible alternatives.

**Definition 1.** Given a travel demand \( g \), a set of possible alternatives \( \{r_1, r_2, ..., r_n\} \) and the set of associated costs \( \{c_1, c_2, ..., c_n\} \) for each alternative, the split factor \( P_{r_i} \), associated with alternative \( r_i \), is a non-negative function \( P_{r_i} \leq 1 \) of \( \{c_1, c_2, ..., c_n\} \) such that \( \sum_i P_{r_i} = 1 \) and the traffic demand \( h_{r_i} \) for alternative \( r_i \), satisfies

\[
h_{r_i} = P_{r_i}(\{c_1, c_2, ..., c_n\}) g.
\]

When the \( c_i \) are deterministic, assuming the users have full information and are rational, the travel demand \( g \) is allocated only on competitive alternatives. In the case of alternatives between two paths \( r_1 \) and \( r_2 \), the split factors are thus:

\[
P_{r_1} = \begin{cases} 
1 & \text{if } c_{r_1} < c_{r_2} \\
0 & \text{if } c_{r_1} > c_{r_2} 
\end{cases}
\]

In order to model the lack of information, or imperfect rationality, the cost \( c_i \) may be seen as a random variable, denoted by \( \tilde{c}_i \). In this case, the split factor is a random variable, for instance:

\[
P_{r_1} = \mathbb{P}(\tilde{c}_{r_1} \leq \tilde{c}_{r_2}).
\]

We call **path choice models** those route choice models under which alternatives are evaluated only at departure nodes, modeling the fact that commuters make a choice only at their departure point, rather than making en-route decisions. For results on equilibria in the context of path choice models, see for instance \([5], [15], [10], [9], [8]\).

We call **adaptive route choice models** those route choice models under which alternatives are evaluated at each node of the network, modeling the fact that commuters update their strategies as they travel on the network, see for instance \([10], [24], [25], [13]\).

### 2.3 Markovian traffic equilibrium

In this section we describe the concept of “Markovian traffic equilibrium” (MTE), introduced in \([3]\), which serves as the underlying model that we extend to the case of non-equilibrium conditions.

#### 2.3.1 Static flow satisfying flow conservation

We denote by \( x_v \) the aggregate demand at node \( v \), defined as the sum of the exogenous traffic demand \( g_v \) and the total entering flow \( \sum_{i \in A_v} f_i \) at node \( v \). Assuming that drivers can switch instantaneously from one arc to another (i.e. that there is no queue or accumulation within a node), we obtain the following flow conservation rule at a node:

\[
\begin{aligned}
f &= Qx \\
x &= g + Rf
\end{aligned}
\]

where \( Q \in \mathbb{R}^{|A| \times |V|}, Q_{i,v} = \delta_{v,o(i)} P_i, \) and \( R \in \mathbb{R}^{|V| \times |A|}, R_{v,i} = \delta_{v,d(i)} \)\(^1\). Figure 2 provides an example of matrices \( R \) and \( Q \) for a simple network and given split factors.

The following definition characterizes a flow respecting the flow conservation rules for a given set of split factors.

\(^1\)with \( \delta_{i,j} \) equals 1 if \( i = j \) and 0 otherwise.
We first write Proof.

**Proposition 1.** Noting \( \tau \) be computed as the solution to a convex programming problem. In particular for a MTE the matrix \( I \) is invertible the static flow is unique and we have

\[
    f = Q(I - RQ)^{-1}g
\]

We shall provide later the necessary and sufficient conditions for the invertibility of the matrix \( I - RQ \).

### 2.3.2 Adaptive routing policy with stochastic travel-times

Our model assumes adaptive route choice, strictly increasing latency functions, and stochastic travel-times. Specifically, the random link-level travel-time \( \tilde{\tau}_a \) is modeled as:

\[
    \tilde{\tau}_a = \tau_a + \epsilon_a \quad \text{where} \quad \epsilon_a \text{ is a white noise with an exogenous standard deviation } \sigma_a .
\]

The cost considered by users at each node of the network is the travel-time to destination when entering the corresponding arc \( a \). It is denoted \( z_a \) and can be computed as:

\[
    \begin{align*}
    z_a &= \tilde{\tau}_a + \min_{j \in A^+_d(a)} \tilde{z}_j \\
    z_a &= \tau_a + \mathbb{E}(\min_{j \in A^+_d(a)} \tilde{z}_j)
    \end{align*}
\]

modelling the fact that each user reaching node \( v \) observes the optimal travel-times \( \tilde{z}_i \) for all \( i \in A^+_v \) and chooses the arc \( i \) with the minimal perceived cost. Therefore the aggregate flow demand \( x_v \) splits into the departing arcs according to the following split factors:

\[
    P_i = \mathbb{P}(\tilde{z}_i \leq \tilde{z}_j, \ \forall j \in \tilde{I}), \quad \forall i \in A^+_v
\]

### 2.3.3 Existence and uniqueness of Markovian Traffic Equilibrium

Considering the model introduced in the previous section, \cite{3} prove that there exists a unique MTE, which can be computed as the solution to a convex programming problem. In particular for a MTE the matrix \( I - RQ \) is invertible.

**Definition 3.** A vector \( f^* \in \mathbb{R}^{|A^*|} \) is a Markovian traffic equilibrium (MTE) if and only if: \( f^* \) satisfies system (1), the split factors used to defined matrix \( Q \) satisfy definition (4), and for all \( a \in A, z_a \) satisfies definition (3) and \( \tau_a = s_a(f^*_a) \).

**Proposition 1.** Noting \( \psi_v(\tau) = \mathbb{E}(\min_{j \in A^+_v} \tilde{z}_j) \), we have \( \nabla_{\tau} \psi = (I - Q^TR^T)^{-1}Q^T \)

**Proof.** We first write \( \psi_v(\tau) \) as a function of \( z \). With \( \tilde{z} = z + \epsilon \) where \( z \) is the vector of the \( z_a, a \in A \), and \( \epsilon \) a vector such that \( \mathbb{E}(\epsilon) = 0 \), if we note \( \phi_v(z) = \mathbb{E}(\min_{j \in A^+_v} (z_j + \epsilon_j)) \) we have \( \psi_v(\tau) = \phi_v(z) \).

\cite{3} prove that for continuous distributions of the \( \epsilon_j \), \( \frac{\partial \phi_v(z)}{\partial z_k} (z) = \mathbb{P}(\tilde{z}_k \leq \tilde{z}_j, \ \forall j \in \tilde{I}) \). Using the expressions (3) and (4) and the fact that \( \phi_v \) depends only on \( j \in A^+_v \) we have:

\[
    \frac{\partial}{\partial \tau_a} \psi_v(\tau) = \sum_{j \in A} \frac{\partial \phi_v}{\partial z_j} \frac{\partial z_j}{\partial \tau_a} = \sum_{j \in A^+_v} P_j \left( \frac{\partial \tau_j}{\partial \tau_a} + \frac{\partial}{\partial \tau_a} \phi_{d(j)} \right) = \sum_{j \in A^+_v} P_j \left( \delta_{a,j} + \frac{\partial}{\partial \tau_a} \psi_{d(j)} \right).
\]

![Figure 2: Example of matrices R and Q for a simple network.](image-url)
Using the definitions of matrices $R$ and $Q$ (1) we recognize in the above expression the component-wise formulation of the following equality: $\nabla_\tau \psi = Q^T(I + R^T \nabla_\tau \psi)$ which concludes the proof.

It follows that

$$
\nabla_\tau (\psi^T g) = (\nabla_\tau (\psi))^T g = Q(I - RQ)^{-1} g = f^*.
$$

(5)

Defining the convex domain $P = \{\tau \in \mathbb{R}^{|A|} : \psi_v(\tau) > 0 \text{ for all } v \in V\}$ [3] prove the following.

**Theorem 1.** For any $\tau^0 \in P$ there exists a unique MTE given by $f_a = s_a^{-1}(\tau_a)$ with $\tau$ the unique optimal solution of the following smooth and strictly convex program

$$
\tau^* = \arg \min_{\tau \in P} \Phi(\tau) = \sum_{a \in A} \int_{\tau_a^L}^{\tau_a} s_a^{-1}(z)dz - \sum_{v \in V} g_v\psi_v(\tau).
$$

(6)

**Proof.** The existence, uniqueness, and concavity of $\psi_v$ are proved using a fixed point algorithm. Equation (5) then concludes the proof.

\[ \square \]

## 3 Lyapunov stability properties of Markovian Traffic Equilibria

In this section we present a dynamical model of non-equilibrium traffic in the context of adaptive routing and present Lyapunov stability results applicable to a generic graph structure.

### 3.1 Dynamical model

In order to study the stability of the equilibrium defined by equation (6) when the network is not at equilibrium, we first introduce a model of traffic dynamics based on flow conservation. We denote the flow entering arc $a$ by $f_a^+$, and the flow leaving arc $a$ by $f_a^-$. Because the network may not be at equilibrium, in our context these two quantities may differ. We denote the traffic volume on arc $a$ (expressed in number of vehicles) by $\rho_a$, and assume that $\rho_a$ is continuous and differentiable with respect to time. Traffic dynamics reads:

$$
\forall a \in A, \forall t > 0, \quad \frac{d\rho_a}{dt}(t) = f_a^+(t) - f_a^-(t),
$$

(7)

where $\rho$ is expressed in terms of number of vehicles and $f$ is expressed in number of vehicles per unit of time. We define the departing flow $f_a^-(t)$ to be a function of the current arc volume $\rho_a(t)$ through a continuous, differentiable and strictly increasing flow function $\mu_a : f_a^- = \mu_a(\rho_a)$, and to be constrained by the capacity $C_a$ of each arc as:

$$
\forall t > 0, \quad f_a^-(t) \leq C_a.
$$

The model captures dynamics within a short time period during which the demand can be considered to be constant, hence the equilibrium is assumed not to change, and an analysis of the stability of small perturbations around that equilibrium can be conducted. Furthermore, flow conservation holds as well as en-route route choice. This model is similar to the dynamic en-route route choice approach of [14] and [10], where the former additionally includes a scaling parameter to control the relative time-scale of flow conservation dynamics compared to route choice dynamics.

We do not model queues and assume that the entering flow $f_a^+(t)$ into an arc is given by the sum of the departing flows from the upstream arcs and the node travel demand. The travel-time $\tau_a$ on arc $a$ is given by a function of the arc volume $\rho_a$ only. We denote $\tau_a = \tau_a(\rho_a)$, so that $\tau_a = \tau_a(\mu_a(\rho))$ is a strictly increasing function of $\rho_a$. We assume that drivers never wait when crossing from one arc to the other (no entry queues). Hence, a system analog to (1) holds where we allow for $f^+ \neq f^-:

$$
\begin{align*}
&f^+ = Qx \\
&x = g + RF^-
\end{align*}
$$

(8)

Thus, using equation (7) we obtain:

$$
\dot{\tau} = D(\tau) = T(\tau)(f^+(\tau) - f^-(\tau)) \quad \text{where } T \in \mathbb{R}^{|A||x|A} \text{ and } T_{i,j} = \delta_{i,j} \frac{d\tau_i}{d\rho_i},
$$

(9)

which shall serve as the main equation for the Lyapunov stability analysis presented in the following section.
### 3.2 Lyapunov theorems

In this section, we summarize definitions of equilibrium and exponential stability, as well as the main Lyapunov theorem used in the following section. The interested reader is referred to [26] for more details. To illustrate the following definitions, consider the non-linear and time-invariant differential equation

\[ \dot{x} = f(x), \quad x(t_0) = x_0, \tag{10} \]

where \( x_0 \) denotes the initial condition, and consider \( x^* \) an equilibrium point of (10) (verifying \( f(x^*) = 0 \)).

**Definition 4.** \( x^* \) is said to be a locally **exponentially stable** equilibrium point of (10) if there exists \( h \) small enough and \( m, \gamma > 0 \) such that

\[ |x(t) - x^*| \leq m |x_0 - x^*| e^{-\gamma t} \]

for all \( x \in B_h(x^*) \) and \( t > 0 \).

The following theorem provides sufficient conditions for an equilibrium \( x^* \) of (10) to be locally exponentially stable.

**Theorem 2.** ([26]) An equilibrium point \( x^* \) of (10) is **locally exponentially stable** if there exists: a differentiable function \( V(x) \), \((\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{R}^4_+ \) and \( h \in \mathbb{R}_+ \) such that \( \forall x \in B_h(x^*) \)

1. \( \alpha_1 |x - x^*|^2 \leq V(x) \leq \alpha_2 |x - x^*|^2 \) \quad \( V \) is locally quadratic
2. \( |\nabla_x V(x)| \leq \alpha_4 |x - x^*| \) \quad \( V \) evolves slowly around \( x^* \)
3. \( (\nabla_x V(x))^T f(x) \leq -\alpha_3 |x - x^*|^2 \) \quad \( \)the system evolves toward \arg\min V(x)\)

The theorem also provides a constant \( m \) and an exponential envelope of convergence \( \gamma \).

\[ m = \sqrt{\frac{\alpha_2}{\alpha_1}}, \quad \gamma = \frac{\alpha_3}{2\alpha_2}. \]

In the next section, we introduce a Lyapunov function candidate and derive stability properties.

### 3.3 Lyapunov analysis

Let us consider the three conditions of Theorem 2.

#### 3.3.1 Lyapunov function candidate

The strictly convex and continuously differentiable function \( (\Phi(\tau) - \Phi(\tau^*)) \) defined in equation (6) is a natural candidate as Lyapunov function. In order to prove the first condition of Theorem 2, we assume that the Hessian of \( \Phi \) is always positive definite\(^2\). Formally, we have the following.

**Lemma 1.** \( \exists (\alpha_1, \alpha_2) \in \mathbb{R}^2_+ \) and \( h \in \mathbb{R}_+ \) such that \( \forall \tau \in B_h(\tau^*) \),

\[ \alpha_1 |\tau - \tau^*|^2 \leq \Phi(\tau) - \Phi(\tau^*) \leq \alpha_2 |\tau - \tau^*|^2. \]

**Proof.** The Hessian matrix \( H_\Phi(\tau^*) \) of \( \Phi(\tau) \) at point \( \tau^* \) is positive definite. If \( \alpha_1 = \min\{\text{EigenValues of } H_\Phi(\tau^*)\} \) and \( \alpha_2 = \max\{\text{EigenValues of } H_\Phi(\tau^*)\} \) then \( \alpha_1 > 0 \), \( \alpha_2 > 0 \) and

\[ \alpha_1 |\tau - \tau^*|^2 \leq (\tau - \tau^*)^T H_\Phi(\tau^*)(\tau - \tau^*) \leq \alpha_2 |\tau - \tau^*|^2. \]

Since \( \nabla_\tau \Phi(\tau^*) = 0 \), the Taylor series of \( \Phi \) at point \( \tau^* \) is given by:

\[ \Phi(\tau) = \Phi(\tau^*) + \frac{1}{2}(\tau - \tau^*)^T H_\Phi(\tau^*)(\tau - \tau^*) + o(|\tau - \tau^*|^2). \]

Therefore, there exists \( h \in \mathbb{R}_+ \) small enough so that, \( \forall \tau \in B_h(\tau^*) \), and Lemma 1 holds. \( \square \)

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\(^2\)In this article, we shall not address the case where this condition is not satisfied.
The result of Lemma 1 proves that the Lyapunov function candidate $\tau \to \Phi(\tau) - \Phi(\tau^*)$ satisfies the first condition of Theorem 2. In order to characterize the conditions under which the second and last inequalities of Theorem 2 hold, we compute the gradient of function $\Phi$ using the conservation equation (8) and the expression (5). In the interest of clarity, in the following derivations, we do not write the dependency on $\tau$.

We remark that equation (5) holds for every state of the traffic network as long as the value $\psi$ is defined according to equation (3). System (8), characterizing the flow conservation at nodes, holds and we have:

$$x = g + Rf^- + RQx - RQx$$
$$x = (I - RQ)^{-1}g + (I - RQ)^{-1}R(f^- - Qx),$$

which implies the following implicit expression of $f^+$ in terms of $R, Q, g$ and $f^-$

$$f^+ = Qx = Q(I - RQ)^{-1}g + Q(I - RQ)^{-1}R(f^- - f^+).$$

We now use equation (5) to write $\nabla_\tau (g^T \psi) = f^+ - Q(I - RQ)^{-1}R(f^- - f^+)$, and we compute $\nabla_\tau \Phi$ using the above expressions of $x$ and $f^+$, as well as $f^- = s^{-1}(\tau)$:

$$\nabla_\tau \Phi = \nabla_\tau \left( \sum_{a \in A} \int_{\tau_a}^{\tau_a^o} s_a^{-1}(z)dz \right) - \nabla_\tau (g^T \psi) = (I + Q(I - RQ)^{-1}R)(f^- - f^+),$$

which expresses the Lyapunov candidate gradient as a linear operator on the difference between entering and leaving flow vectors:

$$\nabla_\tau \Phi = (I + Q(I - RQ)^{-1}R)(f^- - f^+) = (I - QR)^{-1} (f^- - f^+),$$

where the second equality is obtained from the matrix inversion lemma. Using this expression of $\nabla_\tau \Phi$, we may now find a sufficient condition to prove the exponential stability of the dynamical model introduced under adaptive routing. Next, we study the properties of the matrix $(I - QR)^{-1}$.

### 3.3.2 Dual adjacency matrix

In this section we derive properties of the matrix $(I - QR)^{-1}$ needed for the subsequent analysis.

**Definition 5.** Given a tree network with $|A|$ arcs and a set $P = \{P_1, ..., P_{|A|}\} \in \{0, 1\}^{|A|}$, we denote $R \in \mathbb{R}^{|V| \times |A|}$, $R_{v,i} = \delta_{v,d(i)}$ and $Q \in \mathbb{R}^{|A| \times |V|}$, $Q_{i,v} = \delta_{v,o(i)} P_i$. We call “dual adjacency matrix” the matrix $M$ defined by $M = (I - QR)^{-1}$.

The motivation for the label “dual adjacency matrix” comes from the fact that the matrix $(I - QR)^{-1}$ corresponds to the adjacency matrix of the unique graph with edges corresponding to nodes of the traffic network, and nodes corresponding to edges of the traffic network. In the interest of space, this proof is omitted.

**Remark 1.** Using the expression of matrix $M$ from equation (5) (middle expression), we observe that applying matrix $M$ to a unit arc flow vector is equivalent to first mapping the unit arc flow vector to the corresponding node flow vector (by the definition of $R$), computing the induced static arc flow (from equation (2)), and adding it to the initial unit arc flow vector. Hence the terms of matrix $M$ can be interpreted as follows

$$m_{i,j} = \text{static flow on arc } i \text{ resulting from a unit traffic demand on arc } j \{f_j = 1\}$$

Using this interpretation of the matrix $M$, we derive the necessary and sufficient conditions for the invertibility of $I - QR$. The proof is omitted in the interest of space.

**Lemma 2.** Given a graph $G$, the matrix $I - QR$ is invertible if and only if $G$ is acyclic.

We now prove that $|A|$ is an upper bound for the matrix norm $||M||$ derived from the vector norm $|.|$.

**Lemma 3.** Given a graph $G$, the set of split factors $P$ and the matrices $R$ and $Q$ defined in equation (1), the norm of matrix $M$ can be upper bounded as follows

$$||M|| \equiv \sup_{x \in \mathbb{R}^{|A|}} \frac{|Mx|}{|x|} \leq |A|.$$

**Proof.** According to Remark 1, each term in $M$ is less than or equal to 1. The fact that $||M|| \leq \max_{i \in \{1,..,|A|\}} \sum_{j \in \{1,..,|A|\}} m_{i,j}$ concludes the proof.
3.3.3 A sufficient condition for exponential stability

In this section we provide a sufficient condition for the third condition of Theorem 2. We first prove that \( \Phi \) satisfies the second condition of the Lyapunov theorem. For this we require the following lemma on the latency functions, which we state without proof.

Lemma 4. For \( h \in \mathbb{R}_+ \) and for all \( \tau \in B_h(\tau^*) \) we can find \( c_f, C_f \in \mathbb{R}_+^2 \) such that

\[
\forall a \in A, \quad c_f < \frac{1}{s_a^{-1}(\tau_a)} = \frac{df_a^+}{d\tau_a} < C_f.
\]

Proposition 2. The Lyapunov candidate \( \Phi \) satisfies the second condition of the Lyapunov theorem.

Proof. First we note that combining equations (5) and (11), we can express the vector \( f^- - f^+ \) in terms of \( f^- - f^* \) as

\[
f^- - f^* = M(f^- - f^+).
\]

Using Lemma 4, we have \( |M(f^- - f^+)| < C_f |\tau - \tau^*| \) which concludes the proof.

A sufficient condition for the third condition of the Lyapunov theorem to be satisfied is as follows.

Proposition 3. Given a graph \( G \) with a topological order, a split factor set \( \mathcal{P} \) and the associated dual adjacency matrix \( M \). If we have

(i) \( \exists c \in \mathbb{R}_+, \forall \tau, \quad (f^+ - f^-)^T(\tau)(TTM + MT^T)(f^+ - f^-)(\tau) \geq c (f^+ - f^-)^T(f^+ - f^-) \)

then

(ii) \( \exists c \in \mathbb{R}_+, \forall \tau, \quad (\nabla_\tau \Phi)^TD(\tau) \leq -c |\tau - \tau^*|^2 \)

and the third condition of the Lyapunov theorem is satisfied.

Proof. Combining expression (12) and equation (9) we obtain

\[
(\nabla_\tau \Phi)^TD(\tau) = (f^- - f^+)^TMM^T(f^+ - f^-) = -(f^+ - f^-)^T(TTM + MT^T)(f^- - f^+),
\]

hence if condition (i) is satisfied we have

\[
(\nabla_\tau \Phi)^TD(\tau) \leq -\frac{1}{2} c |f^+ - f^-|^2 \leq -c \left( \frac{c_f}{|A|} \right)^2 |\tau - \tau^*|^2
\]

where the second inequality is obtained from Lemma 4 and Lemma 3. This concludes the proof.

We summarize this section with the following result.

Proposition 4. Given a graph \( G \) and the associated dual adjacency matrix \( M(\tau) \). If

\[
\exists \alpha \in \mathbb{R}_+, \forall \tau \in \mathbb{R}_+, \forall x \in \mathbb{R}_+^{|A|}, \quad x^T(T(\tau)M(\tau) + M^T(\tau)T(\tau))x \geq \alpha x^Tx
\] (14)

then the Lyapunov theorem 2 applies and the equilibrium defined in theorem (1) is exponentially stable.

Proof. Lemma 1 and Lemma 3 respectively proved the first and second conditions of Lyapunov theorem. We remark that equation (14) implies assertion (ii) from Proposition 3. Thus under equation (14), the three conditions hold and the Lyapunov theorem applies.

In the next section, we consider a category of network topologies and latency functions for which the sufficient condition of Proposition 4 holds.

4 Technical results for tree networks

The condition of Proposition 4 is not always satisfied for general topologies. We first motivate the focus on a certain category of network topologies for which the condition is satisfied, and we then study the properties of the dual adjacency matrix \( M \) for this type of network.
4.1 Tree networks

Consider Figure 3 which illustrates on the left side the traffic demand on weekdays during the morning peak in Singapore. We wish to derive a simplified version of this network topology that retains the shape of the travel patterns. The three components of the simplified traffic network shown on the right in Figure 3 can be seen as an approximation of the actual travel patterns on the left. They can be modeled as independent from each other in the sense that traffic flow from the east to center does not significantly affect traffic flow from west to center or that flowing from the north to the center. As such a multi-source, single destination network can be seen to be a simplified version of the morning peak traffic in an urban area.

We now provide a formal definition of the tree networks that we shall consider.

Definition 6. A tree network is a directed graph $G = (V, A)$, having a unique sink node $t$, having possibly multiple source nodes and containing no cycle (oriented or not) except for those including the sink $t$. Equivalently, there should be no “merge” node except the sink. A tree network is said to be in Category 1 if its source node $v$ has a single departing arc (i.e. $|A^+_{v}| = 1$) and in Category 2 if it has $n > 1$ departing arcs (i.e. $|A^+_{v}| > 1$).

We call a child arc of $a$ every arc $b$ such as $o(b) = d(a)$. Furthermore, we say that $a$ is upstream of arc $b$ if one can find a set of arcs $a = a_0, a_1, ..., a_n = b$ such that $a_{i+1}$ is a child arc of $a_i$ for all $i \in [0; n - 1]$.

![Figure 4: Tree network examples, Left: Category 1. Right: Category 2.](image)

4.2 Properties of the dual adjacency matrix of a tree network

In this section we focus on the algebraic properties of the dual adjacency matrix of a tree network. In order to prove the condition of Proposition 4, we write the dual adjacency matrices of tree networks in a recursive format. First we prove the following technical lemma.

Lemma 5. If $G$ is a tree network, denoting $P = \{P_1, ..., P_{|A|}\}$ the set of its split factors, the following assertions hold:

- For every arc pair $a$ and $b$ such that $b$ is a direct child of $a$ : $m_{b,a} = P_b$
- For every arc pair $a$ and $b$ such that $a$ is upstream of arc $b$ : $m_{a,b} = 0$
- For every arc pair $a, b, c$ such that arc $a$ is upstream of arc $b$ and $b$ is upstream of arc $c$ : $m_{c,a} = m_{c,b}m_{b,a}$.

Proof. According to Remark 1, $m_{a,b}$ is equal to the static flow induced on arc $a$ by an unit traffic flow on arc $b$ when the split factors are fixed to the set $P$. $G$ being a tree, there is at most one path joining two different arcs (due to the acyclic property of tree networks).
Therefore if \( b \) is a direct child of \( a \), the only influence (in terms of static flow) from \( a \) to \( b \) is through the split factor \( P_b \). Thus in the static equilibrium with fixed \( P \) and a unit traffic flow on arc \( a \), the resulting static flow on arc \( b \) is exactly \( P_b \). This concludes the proof for the first part of the lemma.

The no-cycle property of the tree network \( G \) implies that if arc \( a \) is upstream of arc \( b \), a traffic flow on arc \( b \) does not induce any static flow on arc \( a \) (there exists no directed path from arc \( b \) to arc \( a \)) and the second part of the lemma holds.

The proof of the third part of the lemma is based on the linear property of the static flow described in equation (2) with respect to the demand \( g \). Given three arcs \( a, b, c \) such that there exists a directed path from \( a \) to \( b \) and from \( b \) to \( c \) let us compute the static flow on arc \( c \) induced by an unit static flow demand on arc \( a \). The static flow induced on arc \( c \) by a unit flow on arc \( a \) is \( m_{c,a} \) according to Remark 1. We conclude that the static flow on arc \( c \) induced by a unit traffic flow on arc \( a \), i.e. \( m_{c,a} \), is equal to the product of the two factors \( m_{c,b}m_{b,a} \) which concludes the proof of the third part of the lemma.

We remark that those properties are true only for the topology defined in the previous section. The following Lemma characterizes the recursive nature of dual adjacency matrices for tree networks.

**Lemma 6.** For all sets \( P = \{ P_1, ..., P_{|A|} \} \in [0, 1]^{|A|} \) we have

(i) For all tree networks in Category 1, the dual adjacency matrix is lower triangular and can be written as a block diagonal matrix combined with a non zero first column (reordering the arcs indices):

\[
M = \begin{bmatrix}
1 & 0 & \ldots & 0 \\
p_1M_1^{(1)} & M_1 & 0 & 0 \\
p_2M_2^{(1)} & 0 & M_2 & 0 \\
\vdots & 0 & 0 & \ldots \,
\end{bmatrix}
\]

where \( \{M_i, i \in \{1, \ldots, n\} \} \) are dual adjacency matrices corresponding to smaller Category 1 tree networks, \( M_i^{(1)} \) stands for the first column of matrix \( M_i \), and the sum over the coefficients \( \{p_i, i \in \{1, \ldots, n\} \} \) is equal to 1.

(ii) For all tree networks in Category 2, the dual adjacency matrix can be written as follows

\[
M = \begin{bmatrix}
M_1 & 0 & 0 & 0 \\
0 & M_2 & 0 & 0 \\
0 & 0 & \ldots & 0 \\
0 & 0 & 0 & M_n
\end{bmatrix}
\]

where \( \{M_i, i \in \{1, \ldots, n\} \} \) are the dual adjacency matrices corresponding to smaller Category 1 trees (corresponding to the \( n \) subtrees).

**Proof.** We prove the claim for the two categories by recursion on the tree height:

**Height 1:**

1. Category 1 tree networks are composed of a single arc. Any traffic demand on this arc induces no static flow elsewhere. Thus \( M = I_1 \).

2. Category 2 tree networks are composed of parallel (independent) arcs. Any traffic demand on these arcs induces no static flow on others. This implies that \( M = I_{|A|} \).

**Recursion:**

Let us assume the above property for trees of height strictly inferior to \( H \).
1. We first prove that the property for all smaller trees implies the property for Category 1 trees of height $H$.

Let us consider a height $H$ tree of Category 1, call arc 1 its unique root arc and $n$ the number of children of arc 1. Consider the subtree whose root node is $d(1)$. This extracted network is of height $H-1$ and is in Category 2 (because of Assumption 1, the extracted network cannot be of Category 1). According to the recursion hypothesis, its dual adjacency matrix may be written in the second shape which leads to:

$$M = \begin{bmatrix} m_{1,1} & \cdots & \cdots & \cdots \\ \cdots & M_1 & 0 & 0 \\ \cdots & 0 & M_2 & 0 \\ \cdots & 0 & \cdots & \cdots \\ \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & M_n \end{bmatrix}$$

For every subtree $\text{sub}(i)$, $i \in \{1, \ldots, n\}$ rooted at $d(1)$, $\text{sub}(i)$ is a tree of height $H-1$ of category 1. We note $a(i)$ the index of its first arc and we sort the arcs in $\text{sub}(i)$ from index $a(i)$ to index $a(i+1)-1$ using the topological order restricted to subtree $\text{sub}(i)$ (the resulting global order is still a topological order for the tree). The first line of $M$ consists of zeros according to condition (ii) of Lemma 5. Assertion (i) of Lemma 5 proves that arc $a(i)$ (which is a direct child arc of arc 1) satisfies $m_{a(i),1} = P_{a(i)}$.

For all arcs $b$ in the child-tree $i$ we choose arc $a(i)$ as an intermediary arc between arc 1 and arc $b$ and we use assertion (iii) of Lemma 5 to obtain:

$$\forall b > 1 \ s.t \ b \in \{a(i), \ldots, a(i+1)-1\}, \quad m_{b,1} = m_{b,a(i)}m_{a(i),1} = P_{a(i)}m_{b,a(i)}.$$  

We remark that $(m_{a(i),a(i)}, m_{a(i)+1,a(i)}, \ldots, m_{a(i+1)-1,a(i)})^T = M_i^{(1)}$ which concludes the recursion for Category 1 (noting $p_i \equiv P_{a(i)}$ for convenience).

2. We now prove that the property for all smaller trees induces the property for Category 2 trees. Let us consider a height $H$ tree of Category 2 having $n$ first arcs. For all first arcs $i$, because of Assumption 1, the extracted tree network is of Category 2 and of height $H$. By hypothesis, its corresponding dual adjacency matrix may be written in Category 2 form. Therefore, the dual adjacency matrix of the initial tree network is a block diagonal matrix with each of its submatrices being of Category 2 shape. This concludes the recursion for Category 2 and the proof of Lemma 6.

The result of Lemma 6 allows us to write for all tree networks of Category 1:

$$\forall x = \{x_1, \ldots, x_{|A|}\} \in \mathbb{R}^{|A|}, \quad x^TMx = x_1^2 + x_1 \sum_{i \in \{1, \ldots, n\}} p_i x_{M_i}^T M_i^{(1)} + \sum_{i \in \{1, \ldots, n\}} x_{M_i}^T M_i x_{M_i}$$  \hfill (15)

where $x_{M_i} = (x_{a(i)}, \ldots, x_{a(i+1)-1})$. Lemma 7 provides a lower bound for this quantity in terms of $x^Tx$ for tree network of Category 1. This property is generalized to all tree networks in Proposition 5.

**Lemma 7.** For a tree network of Category 1 with $|A|$ arcs let $M^{(1)}$ denote the first column of the corresponding dual adjacency matrix $M$. Then

$$\forall x \in \mathbb{R}^{|A|}, \quad 2 x^TMx \geq (x^TM^{(1)})^2 + x^Tx.$$  \hfill (16)

**Proof.** We proceed by recursion on the dimension of the Category 1 tree networks, using equation (15).

**Single arc tree** $M = I_1$ and equation (16) is equivalent to $2x^2 \geq x^2 + x^2$ which is true.

**Recursion** Let us consider a Category 1 tree network with $|A|$ arcs and let us assume that equation (16) holds for a tree network of Category 1 with strictly less than $|A|$ arcs. In equation (15), for all $i \in \{1, \ldots, n\}$, $M_i$ corresponds to a tree network of category 1 and dimension strictly less than $|A|$. Therefore using equation (15) we have, with
where the first inequality comes from the recursion hypothesis. Denoting $j = \arg \max_{i \in \{1, \ldots, n\}} |x_{i}^{T}M_{i}^{(1)}|$, we obtain:

$$
2x^{T}Mx - (x^{T}M^{(1)})^2 - x^{T}x \geq \sum_{i \in \{1, \ldots, n\}} (x_{i}^{T}M_{i}^{(1)})^2 - (\max_{i \in \{1, \ldots, n\}} |x_{i}^{T}M_{i}^{(1)}|)^2 \\
\geq \sum_{i \in \{1, \ldots, n\}, i \neq j} (x_{i}^{T}M_{i}^{(1)})^2 \geq 0.
$$

This result yields the more general result for tree networks given by Proposition 5.

**Proposition 5.** For a tree network, noting $M = M(\tau)$ its dual adjacency matrix for a certain state $\tau$,

$$
\forall x \in \mathbb{R}^{|A|}, \quad x^{T}Mx \geq \frac{1}{2}x^{T}x. 
$$

**Proof.** For tree networks of category 1, the result follows from Lemma 7. The result for tree networks of Category 2 comes from the shape of their dual adjacency matrix obtained in Lemma 6. All block diagonal matrices $M_{i}$ composing $M$ correspond to tree networks of Category 1 and therefore satisfy equation (17). The property for $M$ is obtained by decomposing vector $x$ over the block diagonal sub-spaces and applying the property for each matrix $M_{i}$.

5 Exponential stability results for Markovian Traffic Equilibria

Having described the main properties of dual adjacency matrices for tree networks in the previous section, we state in this section our main results using those properties and the sufficient condition from Proposition 4. First we state the main theorem of this article which proves the exponential stability of the equilibrium for tree networks under affine latency functions. We then present an application of the theorem to the independent path choice problem.

5.1 Exponential stability for homogeneous tree networks

We shall now prove that the Markovian Traffic Equilibrium defined in equation (6) is locally exponentially stable under the proposed traffic dynamics and route choice model. Recall that $\tau_{a} = s_{a}(\mu_{a}(\rho_{a}))$ where $s_{a}$ and $\mu_{a}$ are strictly increasing functions and that $s_{a}(0)$ denotes the free flow travel-time on arc $a$. The following assumption yields a sufficient condition on the functions $s_{a}(\mu_{a}(\cdot))$ for the Lyapunov theorem to apply.

**Assumption 2.** **Travel-times on arcs $\tau_{a}$ of $G$ satisfy :** $\exists \theta \in \mathbb{R}_{+}, \forall \tau, \quad T(\tau) = \theta \cdot I_{|A|}$

$$
i e \in \mathbb{R}_{+}, \forall a \in A, \forall \rho_{a} \in \mathbb{R}_{+}, \quad \tau_{a} = \theta \cdot \rho_{a} + s_{a}(0).
$$

This assumption corresponds to the case of close-to-uniform urban networks, in which network links have similar importance with respect to travel patterns.

We may now prove our main result:
Theorem 3. For a tree network satisfying Assumption 2, the MTE $\tau^*$ defined in (6) is locally exponentially stable.

Proof. According to Proposition 5, the dual adjacency matrix satisfies $\forall x \in \mathbb{R}^{|A|}, x^T M x \geq \frac{1}{2} x^T x$. Assumption 2 yields $\exists \theta \in \mathbb{R}_+, \forall \tau, T(\tau) = \theta I_{|A|}$ so that $\forall x \in \mathbb{R}^{|A|},$

$$\forall \tau, x^T (T(\tau) M(\tau) + M^T(\tau) T(\tau)) x = \theta (x^T (M(\tau) + M^T(\tau)) x) \geq \frac{\theta}{2} x^T x.$$ 

Proposition 4 then concludes the proof. \qed

Additionally, we obtain the following expression (see [26]) characterizing the convergence envelope:

$$\exists h \in \mathbb{R}_+ \text{ such that } \forall \tau \in B_h(\tau^*), \forall t > 0, \quad |\tau(t) - \tau^*| \leq |\tau(0) - \tau^*| \sqrt{\frac{\alpha_2}{\alpha_1}} e^{-\frac{s^2}{4m_1^2t}},$$

where $H$ is the height of the tree.

Remark 2. For the case of networks that do not satisfy assumption 2, one may note that only the proof of theorem 3 needs to be adapted, for instance by considering a neighborhood of the equilibrium in which the derivatives of the latency functions are uniformly lower bounded.

5.2 Exponential stability for parallel networks

Consider now the case of parallel networks. In this case, we shall prove a Corollary of theorem 3. We then provide an interpretation of this corollary as a stability result for the MTE in an independent path choice model.

Definition 7. A parallel network is a directed graph $G = (V, A)$ with a unique sink node $t$, a unique source node $s$ and an arbitrary number of arcs directed from $s$ to $t$.

The following corollary gives the exponential stability of the MTE for parallel networks without restriction on the shape of the strictly increasing latency functions.

Corollary 1. For a parallel network with strictly increasing latency functions, the MTE $\tau^*$ defined in equation (6) is locally exponentially stable.

Proof. Let us first remark that the dual adjacency matrix of a parallel network is the identity matrix. This property can be obtained from Remark 1 and has for instance been used in the initialization of the proofs of the dual adjacency shapes for more complex topologies in the previous section. This property implies that $\forall \tau, x^T (T(\tau) M(\tau) + M^T(\tau) T(\tau)) x \geq \min T_{i,i}(\tau) (x^T x)$. Since $T_{i,i} = 0$ is impossible for strictly increasing latency functions, and Proposition 4 then concludes the proof. \qed

Since parallel networks can be used to represent a set of independent path choices from a source node to a sink node, we can interpret Corollary 1 as a result on the stability of an a priori, non-overlapping path choice model.

Remark 3. The problem of drivers choosing a priori (i.e. at their departure node) between independent non-overlapping paths from a unique origin to a unique destination is formally equivalent to the problem discussed above. Therefore Corollary 1 allows us to state that this problem is locally exponentially stable for any given strictly increasing arc latency functions.

6 Numerical results

In this section we present the results of our numerical experiments. We compute a numerical approximation to the continuous time traffic model and use it to illustrate that the numerical sensitivity of the characteristic convergence time to the shape of latency functions and network topologies corresponds to the theoretical sensitivity predicted by our theoretical results. We also present numerical insights regarding comparative stability properties of online adaptive route choice and a-priori path choice. Finally, we illustrate the applicability of our theoretical results using a real network by simulating the stability properties of a subset of the Singapore road network, using observed traffic demands and road network properties.
6.1 Numerical scheme

In this section we present the numerical discretization of the continuous time dynamical model of section 3. The discretized time is indexed by $k \in \mathbb{N}$, and the numerical approximation of the traffic volume and flow at time $k$ reads:

\[
\text{Step 1:} \quad \forall a \in A, \quad \rho^k_a = \rho^{k-1}_a + f^{(k-1)}_a - f^{(k-1)}_a
\]

\[
\text{Step 2:} \quad \forall a \in A, \quad f^{(k)}_a = \mu_a(\rho^k_a)
\]

\[
\forall a \in A, \quad \tau^k_a = s_a(f^{(k)}_a)
\]

The split factors $P^k_a$ are computed from the latency values $\tau^k_a$ using a Monte Carlo method. We first add independent gaussian noise to each latency so that $\tilde{\tau}_a^k = \tau_a^k + \epsilon_a$ with $\mathbb{E}(\epsilon_a) = 0$ and $\mathbb{E}(\epsilon_a^2) = \sigma^2$. We then compute the travel time to destination $\tilde{z}_a^k$ along the shortest path. We repeat the above sequence $N$ times ($N = 1000$ in the simulation) and compute the split factors $P^k_a$ as the proportion of the iterations where $\tilde{z}_a^k < \tilde{z}_b^k \forall b \in o(a)^+$.

6.2 Sensitivity of characteristic convergence time to traffic parameters

In this section we study (for the case of exponential stability) the numerical dependency of the characteristic convergence time derived in section 5.1 to two meaningful traffic parameters; we first study sensitivity to the slope of the latency function, characterizing the sensitivity of travel-time to volume, and then the sensitivity to the intercept of the latency function, characterizing the free flow travel-time.

6.2.1 Sensitivity of characteristic convergence time to derivative of latency function

We note $\gamma$ the characteristic convergence time of the system to the MTE. It follows from the expression of the exponential convergence envelope that the expression of $\gamma$ is:

\[
\gamma = \frac{4H^2\alpha_2}{\theta \epsilon_f^2}.
\] (19)

In order to characterize the analytical dependency of $\gamma$ to the latency function parameters, we first recall the shape of the latency function in the scenario of Theorem 3, $\tau_a = \theta \rho_a + \beta_a$, and we remark that in expression (19), $\theta$ appears both explicitly and implicitly through the factors $c_f$ and $\alpha_2$. The conservation of volume $\rho_a = f_a^* \tau_a$ combined with Assumption 2 induces that $c_f$ depends on $\theta$ as follows

\[
c_f = \min_{a \in A, \rho_a > 0} \frac{d f_a^*}{d \tau_a}(\rho_a) = \min_{a \in A, \rho_a > 0} \left( \frac{\beta_a}{\theta \tau_a^2} \right) = \frac{1}{\theta} \min_{a \in A, \rho_a > 0} \left( \frac{\beta_a}{(\theta \rho_a + \beta_a)^2} \right).
\] (20)

This expression implies $c_f \in \Theta(\theta^{-3})$ when $\theta \gg \frac{\rho_o}{\epsilon_f}$, i.e. that $c_f$ evolves with $\theta^{-3}$ for $\theta$ large enough. As described in Proposition 1, $\alpha_2$ can be computed from the Hessian of $\Phi$ in $\tau = \tau^*$ noted $H_\Phi(\tau^*)$. The shape of the Hessian $H_\Phi(\tau^*)$ is given in [3] as

\[
H_\Phi(\tau^*) = \text{diag}(\frac{d f_a^*}{d \tau_a}(\rho_a)) - J(\tau^*)
\]

where $J(\tau^*)$ depends on $g$, the split factors at equilibrium and their derivative at equilibrium. It is therefore reasonable to assume that the eigenvalues of $H_\Phi(\tau^*)$ have the same type of dependency with respect to $\theta$ as $c_f$. In particular, we expect $\alpha_2 = \max\{\text{eigenvalues of } H_\Phi(\tau^*)\}$ to evolve as $\theta^3$ for $\theta$ big enough. Assuming this property for $\alpha_2$ we deduce from equation (19) the following theoretical dependency of $\gamma$ for $\theta$ large enough:

\[
\gamma_{\text{the.}} \in \Theta(\theta^2).
\]

In order to validate this dependency, we consider a tree network of height 4 and degree 4, with link latency functions affine of slope $\theta$. For different initial conditions, we observe the numerical convergence time and compute the numerical value of the characteristic convergence time as follows:
Routine for characteristic convergence time numerical estimation

Step 1: Compute the equilibrium volumes on the arcs, $\rho^*$, using equation (18). Define $\rho^* = \rho^K$ where $K$ is the first time step for which $|\rho_{k+1} - \rho_k| < \epsilon_1|\rho_k|$. 

Step 2: Initialize conditions to randomly chosen initial volumes on arcs. 

Step 3: Store the “estimated characteristic time” as the value of $k$ such that $|\rho_k - \rho^*| < \epsilon_2|\rho_0 - \rho^*|$ with the scaling $\gamma = k \ln(\epsilon_2^{-1})^{-1}$ so that the result is independent of the choices of $\epsilon_1$ and $\epsilon_2$. 

Step 4: Return the average value of the “estimated characteristic times” obtained for ten iterations of steps 2 and 3. 

The numerical dependency of the characteristic convergence time, illustrated in Figure 5, is consistent with the theory that suggested a quadratic dependency. This result was verified to be independent from any particular tree topology by testing a large number of different tree topologies. For typical convex latency functions, this local analysis illustrates that the closer the network is to approaching capacity, the larger the derivative of the latency function, hence the larger (quadratically) the convergence time, confirming the significant non-linear impact of congestion on road usage. 

6.2.2 Sensitivity of characteristic convergence time to free-flow travel time 

In this section, we analyze the sensitivity of the characteristic convergence time (as defined by equation (19)) to the free flow travel-time on arcs. First we recall that the free flow travel-time on arc $a$ is given by $s_a(0) = \tau_a(f_a = 0) = \tau_a(\rho_a = 0) = \beta_a$. 

We consider a tree network for which the travel-time depends on volume in an affine manner, and uniformly across all links, such that for all $a \in A$, $\tau_a = \theta \rho_a + \beta$. In this context, Theorem 3 applies and the characteristic convergence time is given by equation (19). Equation (20) from the previous section becomes 

$$c_f = \frac{\beta}{\theta} \min_{a \in A, \rho_a > 0} \left( \frac{1}{(\theta \rho_a + \beta)^2} \right),$$

so that $c_f \in \Theta(\beta^{-1})$ when $\beta \gg \rho_a \theta$. We recall from the previous section that we expect the eigenvalues of $H_{\Phi}(\tau^*)$ to behave as $c_f$ for high enough values of $\beta$. This remark leads to the following theoretical dependency: 

$$\gamma_{\text{the}} \in \Theta(\beta).$$

The numerical characteristic convergence time is computed similarly to previous section, and illustrated in Figure 6. The numerical sensitivity of the characteristic convergence time is coherent with its theoretical linear dependency. We verified empirically that this result was independent of the tree topology by iterating the above routine over many tree topologies. This sensitivity can be physically interpreted as the fact that the shorter the link lengths, the faster the convergence to the equilibrium.
6.3 Comparative analysis of convergence time in online route choice and a-priori route choice

In this section we compare the numerical stability properties of two route choice models, online route choice and a-priori route choice. We consider two small networks with similar topology, presented in Figure 7.

In order to compare the stability properties of the two equivalent networks we allow for different equilibria by imposing the following condition on the travel-time functions:

\[ T_A = T_B = \theta I_4, \]

which induces, using the symmetry of each of the topologies, that at equilibrium the flow splits exactly in half each time it is allowed to. This implies:

\[ \tau^*_A = \tau^*_B = \tau^*_A = \tau^*_B = \tau^*_A = \tau^*_B = \tau^*_A = \tau^*_B. \]

We now compare the stability of these two equilibria \( \tau^*_A \) and \( \tau^*_B \) via their characteristic convergence time. We set initial volumes of the four arcs to random values (we set the volumes of corresponding arcs of the two topologies to the same value randomly chosen), simulate the traffic evolution and compute the characteristic convergence time as in Step 3 of the routine described in the previous section. This process is repeated over 1000 initial conditions and the distribution of the difference between the two characteristic convergence time \( (\gamma_A - \gamma_B) \) is presented in Figure 8.

We remark that network \( A \) converges always at least as fast as network \( B \). The equilibrium defined by the convex programming problem (6) can be seen as the Nash equilibrium of a traffic network where users may update their path choice online. Network \( A \) is obtained from network \( B \) by providing an additional degree of freedom to users (allowing them to update their choice in node \( v \)). Unlike for the Braess paradox classically illustrated on similar networks, the addition of a degree of freedom impacts “positively” the stability property of the network in the sense of reducing its convergence time towards equilibrium.

6.4 Analysis of Singapore road network

In this section we present a numerical experiment on the case of the Singapore traffic network. Similar to section 4.1, we use a simplified network induced by the demand zones in the eastern part of Singapore. The main origin and destination zones are estimated from publicly available data, which yields the simplified topology presented in Figure 9. We further decompose the network into 4 zones A, B, C, D, between which most trips occur. Using
Distribution of the difference in convergence time \( \gamma_A - \gamma_B \) over 1000 initial conditions for \( \gamma_B \approx 50 \).

Figure 9: Simplified network of eastern Singapore: main origin zones in green, main destination zone in red.

publicly available data, we scale the demand to match observed flow levels on roads in the morning, and we use Google Maps to estimate free flow travel-times between zones. These parameters, displayed in Figure 10, are used to calibrate a latency function model.

The time evolution of this simplified network for random initial volumes around the equilibrium is displayed in Figure 11. We observe that all the volumes converge toward their equilibrium value. Moreover this equilibrium seems to be unique (as predicted theoretically) since all the random volumes tested induce the same final traffic state. We also remark that, as expected, the equilibrium volume level on the path of smaller free flow travel-time is higher. Furthermore, we notice the particularly low volume level of road 6. This is explained by the larger difference of free flow travel-times \( (s_a(0)) \) between the competitor arcs 5 and 6. The convergence observed is not always monotone (e.g road 5), but seems to lie in an exponential envelope as forecasted by our theoretical results. In particular, using the routine described in previous section, we compute a numerical characteristic convergence time of 12 minutes for this network. This time may be seen as an indicator of the time needed for the partitioning of traffic demand between possible paths to stabilize at the equilibrium when users adapt their choice on the network using real time information. This information is critical to real-time traffic operations. Indeed, while more connected rational users having faster access to real-time information yields a more stable network, allowing for appropriate reaction time to the network once a congestion mitigation measure is implemented is still required.

7 Conclusion

In this paper, we extended the framework of “Markovian Traffic Equilibria” to handle perturbations around traffic equilibria when users update their route choice at each node of the network based on traffic conditions and traffic stochasticity. We showed that for tree networks with locally affine uniform link-level latency functions, any MTE is exponentially stable. We illustrated the theoretical results introduced on toy networks as well as on a real network with field data from Singapore, in particular we validated the analytical expression (value and sensitivity) of the characteristic time of convergence. Valuable extensions to this work include the analysis of controllability properties.
<table>
<thead>
<tr>
<th>Road name</th>
<th>label</th>
<th>$s_a(0)$</th>
<th>$C_a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>East coast road 1</td>
<td>1</td>
<td>4</td>
<td>150</td>
</tr>
<tr>
<td>East coast road 2</td>
<td>2</td>
<td>6</td>
<td>150</td>
</tr>
<tr>
<td>Geyland road 1</td>
<td>3</td>
<td>7</td>
<td>150</td>
</tr>
<tr>
<td>Geyland road 2</td>
<td>4</td>
<td>6</td>
<td>150</td>
</tr>
<tr>
<td>Guillemard</td>
<td>5</td>
<td>6</td>
<td>60</td>
</tr>
<tr>
<td>Serangoon</td>
<td>6</td>
<td>8</td>
<td>90</td>
</tr>
<tr>
<td>Mountbatten road</td>
<td>7</td>
<td>9</td>
<td>60</td>
</tr>
<tr>
<td>Tanjong Katong road</td>
<td>8</td>
<td>8</td>
<td>60</td>
</tr>
</tbody>
</table>

Figure 10: **Simplified network of eastern Singapore, Left:** free-flow travel-time in minutes, capacity in vehicles per minute, **Right:** network structure. $s_a$ is expressed in minutes and $C_a$ in vehicles per minute.

Figure 11: **Simulated evolution of link-level traffic volume** under an adaptive routing policy for the network properties described in figure 10.

of transportation networks under this and other adaptive online route choice models.

References


