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CLOSEDNESS AND NORMAL SOLVABILITY OF AN OPERATOR GENERATED BY A DEGENERATE LINEAR DIFFERENTIAL EQUATION WITH VARIABLE COEFFICIENTS

S. M. Zhuk[1]

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For a linear operator $\mathcal{D}\colon \mathbb{W}_2^F \subset \mathbb{L}_2^n \to \mathbb{L}_2^m \times \mathbb{R}^m$ generated by the differential equation

$$\frac{d}{dt}Fx(t) - C(t)x = f(t), \quad Fx(t_0) = f_0,$$

we prove that its graph is closed and determine the adjoint operator $\mathcal{D}^*\colon \mathbb{W}_2^{F'} \subset \mathbb{L}_2^m \times \mathbb{R}^m \to \mathbb{L}_2^n$. For elements of the linear manifolds $\mathbb{W}_2^F$ and $\mathbb{W}_2^{F'}$, we propose an analog of the formula of integration by parts. We establish a criterion for the existence of a pseudosolution of the operator equation $\mathcal{D}x(\cdot) = (f(\cdot), f_0)$ and formulate sufficient conditions for the normal solvability of the operator $\mathcal{D}$ in terms of relations for blocks of the matrix $C(t)$. The results obtained are illustrated by examples.

1. Introduction

In the process of solution of various problems in applied fields such as mathematical economics, robotics, biotechnology, digital image processing, control theory, circuit theory, radiophysics, and chemical and biological kinetics, researchers encounter systems whose state is described by differential equations with degeneracy:

$$F(t)\dot{x}(t) + C(t)x(t) + B(t)f(t) = 0. \tag{1}$$

In literature, systems of the form (1) are called degenerate [1], algebraic–differential [2], singular [3–5], descriptor[2] [6–9], and implicit [10] or unresolved with respect to the derivative [11].

For autonomous systems (1) in the case where the pencil of matrices $\lambda F + C$ is regular,[3] the notion of central canonical form proposed in [5] proved to be fruitful. Namely, for a smooth vector function $f(\cdot)$, a stationary regular system can be reduced to the following form by a nondegenerate linear transformation [7, p. 14]:

$$\dot{x}_1(t) = Ax_1(t) + Kf(t), \quad x_2(t) = -Df(t) - \sum_{i=1}^{m-1} N^i Dv^{(i)}(t)), \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = Q^{-1}x(t).$$

Sufficient conditions for the reducibility of (1) to a central canonical form were proposed in [1]. Note that the theory of singular matrix pencils was used as early as in [12, p. 348] for the description of the set of solutions of stationary singular linear differential equations.

[1] Shevchenko Kyiv National University, Kyiv.

[2] In the present paper, we use the term "descriptor equation" or "descriptor system" to denote systems of the type (2). The emerging differences are described in what follows (see Remark 1).

[3] A pencil of matrices $\lambda F + C$ is called regular if $\det(\lambda_0 F + C) \neq 0$ for a certain real λ_0.

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In the case of variable coefficients, the notion of central canonical form gives information on the structure of a general solution of the system only in special cases, e.g., [2, p. 8] for

$$\mathrm{rank} F(t) = \deg\det(\lambda F(t) + C(t)) = \mathrm{const}, \quad t \in [t_0, T].$$

In [2, 3], the problem of the existence of solutions of initial- and boundary-value problems for Eqs. (1) was studied on the basis of the notion of left (right) regularizing operator [2]:

$$\Lambda_{*,r}[F(t)\dot{x}(t) + C(t)x(t) + f(t)] = \dot{x}(t) + \Lambda_{*,r}[C(t)]x(t) + \Lambda_{*,r}[f(t)],$$

where

$$\Lambda_{*,r} = \sum_{j=0}^{r} L_j(t) \left(\frac{d}{dt}\right)^i.$$

The existence of a left regularizing operator for autonomous systems [2, p. 11] is equivalent to the existence of a central canonical form. In the case of variable coefficients, conditions for the existence of a left regularizing operator [2] are connected with properties of the "extended" system (1), i.e., the system that consists of Eq. (1) and its total time derivatives up to a certain order.

Attention of some investigators is focused on the study of qualitative problems in the theory of descriptor systems with operator coefficients in Banach spaces. The behavior of a descriptor system with delay in a Banach space was investigated in [13] on the basis of analysis of poles of the resolvent of the operator pencil $R(t,\lambda) = (\lambda F(t) + C(t))^{-1}$.

For detailed information on problems in the theory of descriptor systems such as the existence and uniqueness of solutions, construction of numerical methods, control and observability, see [6, 8].

2. Statement of the Problem

In most papers presented above, system (1) was studied on the basis of notions of central canonical form and regularizing operators, i.e., the construction of solutions was based on the reduction of Eq. (1) to a certain canonical form, which enables one to use methods developed for normal ordinary differential equations. In this case, the structure of the corresponding descriptor equation is restricted.

In the present paper, we use an alternative approach based on the investigation of the properties of an operator generated by the linear descriptor equation

$$\frac{d}{dt}Fx(t) - C(t)x(t) = f(t),$$
$$Fx(t_0) = f_0 \tag{2}$$

without assuming the possibility of its reduction to a certain canonical form. Operator methods enable one to use general properties of a broad class of descriptor systems, including so-called "noncausal" systems (see [6, 10]).

In Eq. (2), let F be an $m \times n$ matrix, let $t \mapsto C(t)$ be a continuous $m \times n$ matrix-valued function, and let

$$f(\cdot) \in \mathbb{L}_2([t_0, T], \mathbb{R}^m) := \mathbb{L}_2^m, \quad -\infty < t_0 < T < +\infty, \quad f_0 \in \mathbb{R}^m.$$

It is known that, in the case $F = E,$ a vector function $x(\cdot) \in \mathbb{L}_2^n$ that satisfies the Volterra integral equation of the second kind

$$x(t) = f_0 + \int\limits_{t_0}^{t} (C(s)x(s) + f(s))ds$$

is a solution of Eq. (2). An equation of this type has a unique solution $x(\cdot)$ in the class of absolutely continuous functions, and, furthermore, $x(\cdot)$ satisfies Eq. (2) almost everywhere.

Let us clarify what we mean by a solution of (2) in the case of an arbitrary constant rectangular matrix $F = \{F_{ij}\}_{1,1}^{m,n}$. We introduce the set

$$\mathbb{W}_2^F := \{x(\cdot) \in \mathbb{L}_2^n : Fx(\cdot) \in \mathbb{W}_2^m\},$$

where $\mathbb{W}_2^m$ denotes the collection of all absolutely continuous vector functions from $\mathbb{L}_2^m$ whose derivatives lie in $\mathbb{L}_2^m$, and Fx is an m-vector function whose ith component is a linear combination of components of $x(\cdot)$ with coefficients $\{F_{ij}\}_{j=1}^n$. The linear set $\mathbb{W}_2^F$ is everywhere dense in $\mathbb{L}_2^n$ because $\mathbb{W}_2^n \subset \mathbb{W}_2^F$. By analogy, we introduce the set

$$\mathbb{W}_2^{F'} := \{z(\cdot) \in \mathbb{L}_2^m : F'z(\cdot) \in \mathbb{W}_2^n\}.$$

We define a linear operator $\mathcal{D}$ by the relations

$$\mathcal{D}x(\cdot) = \left(\frac{d}{dt}Fx(\cdot) - Cx(\cdot), Fx(t_0)\right), \quad x(\cdot) \in \mathscr{D}(\mathcal{D}) := \mathbb{W}_2^F,$$

where $\dfrac{d}{dt}Fx(\cdot) \in \mathbb{L}_2^m$ is the derivative of the vector function $Fx(\cdot),$ $Cx(\cdot) \in \mathbb{L}_2^m$ is a vector function $t \mapsto C(t)x(t),$ and $Fx(t_0)$ is the value[4] of $t \mapsto Fx(t)$ for $t = t_0$. Note that, for $x(\cdot) \in \mathbb{W}_2^F$, the equality $\dfrac{d}{dt}Fx(t) = F\dot{x}(t)$ can be absent.

In the case of an arbitrary constant rectangular matrix $F,$ a solution of (2) is understood as an element of the set $\mathbb{W}_2^F$ that satisfies the operator equation

$$\mathcal{D}x(\cdot) = (f(\cdot), f_0).$$

The operator $\mathcal{D}$ is defined in a similar way as in S. Krein's work [14] for boundary-value problems for linear differential equations of order $n.$ It follows from the structure of the introduced operator equation that the vector function $x(\cdot)$ from the space $\mathbb{L}_2^n$ belongs to the set of solutions of (2) for the fixed initial condition f_0 and right-hand side $f(\cdot)$ if the vector function $Fx(\cdot)$ is absolutely continuous and has the derivative of the class $\mathbb{L}_2^m$ satisfying the first equality in (2) almost everywhere and, furthermore, the second equality in (2) is true. Descriptor equations of the form (2) were considered in [9, 11].

The aim of the present paper is to investigate properties of the operator $\mathcal{D},$ namely, its closedness and normal solvability. In terms of descriptor systems, we investigate the problem of the generalized solvability of Eq. (2), conditions for the continuous (in the metric of the corresponding Hilbert space) dependence of a pseudosolution on the right-hand side and initial condition, and the approximation of solutions by elements of a regularizing sequence.

[4] The symbol $Fx(t)$ makes sense for any $t \in [t_0, T]$ because, by virtue of the inclusion $x(\cdot) \in \mathbb{W}_2^F,$ the vector function $t \mapsto Fx(t)$ is absolutely continuous, which, in the general case, cannot be stated for $x(\cdot)$.

3. Closedness of the Operator $\mathcal{D}$. Form of the Adjoint Operator $\mathcal{D}^*$

Generally speaking, the introduced operator $\mathcal{D}$ is not a Fredholm one (Example 3), and, moreover, it is not even normally solvable (Example 2) unlike the operator of ordinary differentiation in $\mathbb{L}_2^n$. However, in the general case, it is densely defined and closed (Theorem 1).

Theorem 1. *For $x(\cdot) \in \mathbb{W}_2^F$ and $z(\cdot) \in \mathbb{W}_2^{F'}$, the following analog of the formula of integration by parts is true:*

$$\int\limits_{t_0}^{T} \left(\left(\frac{d}{dt}\, Fx(t), z(t) \right) + \left(\frac{d}{dt}\, F'z(t), x(t) \right) \right) dt = (Fx(T), F'^{+}F'z(T)) - (Fx(t_0), F'^{+}F'z(t_0)). \tag{3}$$

The operator $\mathcal{D}$ is closed, and its adjoint $\mathcal{D}^*$ is determined by the relations

$$\mathcal{D}^*(z(\cdot), z_0) = L(z(\cdot), z_0) := -\frac{d}{dt}\, F'z(\cdot) - C'z(\cdot),$$

$$\mathscr{D}(\mathcal{D}^*) = \mathscr{D}(L) \; := \; \{(z(\cdot), F'^{+}F'z(t_0) + d) : z(\cdot) \in \mathbb{W}_2^{F'}, F'z(T) = 0, F'd = 0\}.$$

Proof. We choose $x(\cdot) \in \mathbb{W}_2^F$ and $z(\cdot) \in \mathbb{W}_2^{F'}$ and use the formula of integration by parts for the absolutely continuous vector functions $F^{+}Fx(\cdot)$ and $F'z(\cdot)$. We obtain

$$\int\limits_{t_0}^{T} \left(\left(\frac{d}{dt} F^{+}Fx(t), F'z(t) \right) + \left(F^{+}Fx(t), \frac{d}{dt} F'z(t) \right) \right) dt = \big(F^{+}Fx(T), F'z(T)\big) - \big(F^{+}Fx(t_0), F'z(t_0)\big) .$$

By virtue of the Moore–Penrose theorem, we get $FF^{+}F = F$. Therefore, almost everywhere, we have

$$F \frac{d}{dt} F^{+}Fx(t) = \frac{d}{dt}\, FF^{+}Fx(t) = \frac{d}{dt}\, Fx(t),$$

whence

$$\int\limits_{t_0}^{T} \left(\frac{d}{dt}\, F^{+}Fx(t), F'z(t) \right) dt = \int\limits_{t_0}^{T} \left(F\, \frac{d}{dt}\, F^{+}Fx(t), z(t) \right) dt = \int\limits_{t_0}^{T} \left(\frac{d}{dt}\, Fx(t), z(t) \right) dt$$

for every absolutely continuous $z \in \mathbb{L}_2^m$ $\big($and, hence, for any $z \in \mathbb{L}_2^m\big)$.

We prove the theorem in the case[5] $C(t) \equiv 0$. Let us show the closedness of $\mathcal{D}$. To this end, we note that the operator $x(\cdot) \mapsto \dfrac{d}{dt}\, Fx(\cdot)$, $x(\cdot) \in \mathbb{W}_2^F$, is closed.

[5] A generalization is carried out by standard reasoning on the closedness of the sum of closed and bounded operators and on the operator adjoint to this sum.

Indeed, if $x_n(\cdot) \to x$ and $\dfrac{d}{dt}Fx_n(\cdot) \to z(\cdot)$,[6] then $Fx_n(\cdot)$ is absolutely continuous and $Fx_n(\cdot) \to Fx(\cdot)$. Taking into account that the operator of differentiation is closed in $\mathbb{L}_2^m$, we establish that $Fx(\cdot)$ is absolutely continuous and $\dfrac{d}{dt}Fx(\cdot) = z(\cdot)$.

Let $\mathcal{D}x_n(\cdot) \to (z(\cdot), z_0)$ and $x_n(\cdot) \to x(\cdot)$. Then $x(\cdot) \in \mathbb{W}_2^F$, $\dfrac{d}{dt}Fx(\cdot) = z(\cdot)$, and, according to the results proved above, it remains to show that $z_0 = Fx(t_0)$. Indeed, by assumption, we have $Fx_n(t_0) \to z_0$. On the other hand[7]

$$\|Fx(t_0) - Fx_n(t_0)\| \le J\|Fx_n(\cdot) - Fx(\cdot)\|_2 + \sup_{t\in[t_0,c]}\left\|\int_{t_0}^{t}\left(\frac{d}{dt}Fx(s) - \frac{d}{dt}Fx_n(s)\right)ds\right\| \to 0,$$

where $J := (c - t_0)^{-\frac{1}{2}}$.

We have shown the closedness of the densely defined operator $\mathcal{D}$, which proves the existence of $\mathcal{D}^*$ [15, p. 40]. Let us show that the other statements are also true.

Let $(z(\cdot), z_0) \in \mathscr{D}(L)$ and $x(\cdot) \in \mathscr{D}(\mathcal{D})$. Then

$$\langle \mathcal{D}x(\cdot), (z(\cdot), z_0)\rangle_1 = \int_{t_0}^{T}\left(\frac{d}{dt}Fx(t), z(t)\right)dt + \big(Fx(t_0), F'^+F'z(t_0)\big)$$

$$= (Fx(T), F'^+F'z(T)) - \int_{t_0}^{T}\left(\frac{d}{dt}F'z(t), x(t)\right)dt = \langle L(z(\cdot), z_0), x(\cdot)\rangle_2$$

and, according to (3), $L \subset \mathcal{D}^*$. It remains to prove that $\mathscr{D}(\mathcal{D}^*) \subset \mathscr{D}(L)$. Indeed, let $v(\cdot) = \mathcal{D}^*(z(\cdot), z_0)$ for a certain $(z(\cdot), z_0) \in \mathscr{D}(\mathcal{D}^*)$. Then $v(\cdot) \perp \mathcal{N}(\mathcal{D})$. Since

$$F(E - F^+F) = 0 \;\Rightarrow\; \left(\frac{d}{dt}F(E - F^+F)x(\cdot), F(E - F^+F)x(t_0)\right) = (0, 0) \quad \forall x(\cdot) \in \mathbb{W}_2^n,$$

we conclude that $v(\cdot) = F^+Fv(\cdot)$ almost everywhere. In this case,

$$\int_{t_0}^{T}\left(\frac{d}{dt}Fx(t), z(t)\right)dt = \int_{t_0}^{T}\big(F'^+v(t), Fx(t)\big)\,dt = \int_{t_0}^{T}\left(\int_{t}^{T}F'^+v(s)ds, \frac{d}{dt}Fx(t)\right)dt$$

for all $x(\cdot) \in M := \{x(\cdot) \in \mathbb{W}_2^F : Fx(t_0) = 0\}$, and, hence,

$$z(\cdot) - \int_{t}^{T}F'^+v(s)ds \perp \mathcal{R}(L_0),$$

[6] Convergence is understood in the sense of the norm $\|\cdot\|_2 := \langle\cdot,\cdot\rangle_2^{1/2}$ of the space $\mathbb{L}_2^n$.

[7] The quantity $\|\cdot\|$ is the norm in the space $\mathbb{R}^n$.

where L_0 denotes the operator $x(\cdot) \mapsto \frac{d}{dt} Fx(\cdot),\ x(\cdot) \in M.$ If

$$g(t) = \int_{t_0}^{t} FF^{+}\varphi(s)ds, \quad \varphi(\cdot) \in \mathbb{L}_2^m,$$

then $L_0 g(\cdot) = FF^{+}\varphi(\cdot)$ and, hence, $FF^{+}\varphi(\cdot) \in \mathcal{R}(L_0)$ for any $\varphi(\cdot) \in \mathbb{L}_2^m,$ which yields

$$\int_{t_0}^{T} \left(z(t) - \int_{t}^{T} F'^{+}v(s)ds, FF^{+}\varphi(t) \right) dt = 0 \quad \forall \varphi(\cdot) \in \mathbb{L}_2^m.$$

In this case, we have

$$F' \left(z(t) - \int_{t}^{T} F'^{+}v(s)ds \right) = F'(E - F'^{+}F') \left(z(t) - \int_{t}^{T} F'^{+}v(s)ds \right) = 0,$$

i.e., $z(\cdot) \in \mathbb{W}_2^{F'}$ and $F'z(T) = 0.$ Thus, for any $x(\cdot) \in \mathbb{W}_2^F,$ we get

$$\int_{t_0}^{T} \left(\left(\frac{d}{dt} Fx(t), z(t) \right) + \left(\frac{d}{dt} F'z(t), x(t) \right) \right) dt = -(Fx(t_0), F'^{+}F'z(t_0))$$

by virtue of (3). On the other hand,

$$\int_{t_0}^{T} \left(\frac{d}{dt} Fx(t), z(t) \right) dt + (Fx(t_0), z_0) = \int_{t_0}^{T} (\mathcal{D}^*(z(\cdot), z_0)(t), x(t)) dt$$

and, hence,

$$(Fx(t_0), z_0 - F'^{+}F'z(t_0)) = \int_{t_0}^{T} \left(\mathcal{D}^*(z(\cdot), z_0)(t) + \frac{d}{dt} Fx(t), x(t) \right) dt$$

for all $x(\cdot) \in \mathbb{W}_2^F$ and, in particular, for $x(\cdot) \in M$ (see the definition of M given above). Taking into account that $\operatorname{cl} M = \mathbb{L}_2^n,$ we obtain

$$(Fx(t_0), z_0 - F'^{+}F'z(t_0)) = 0$$

for all $x(\cdot) \in \mathbb{W}_2^F.$ It is now clear that $z_0 = F'^{+}F'z(t_0) + d$ and $F'd = 0.$

The theorem is proved.

Remark 1. Generally speaking (see Example 1), the operator $x(\cdot) \mapsto F\dfrac{d}{dt}\,x(\cdot),\;\; x(\cdot) \in \mathbb{W}_2^n,$ is not closed. In particular, this means that

$$F'\frac{d}{dt} \subset \left(\frac{d}{dt}F\right)^*.$$

On the other hand, the space $\mathbb{W}_2^n$ is not always a Hilbert space with respect to the norm of the graph of $\dfrac{d}{dt}\,F.$ Indeed, if $\mathbb{W}_2^n$ is a Hilbert space, then the restriction of $\dfrac{d}{dt}F$ to $\mathbb{W}_2^n$ is a closed operator [15, p. 26]. However,

$$\frac{d}{dt}Fx(\cdot) = F\frac{d}{dt}\,x(\cdot) \quad \forall x(\cdot) \in \mathbb{W}_2^n,$$

i.e., $F\dfrac{d}{dt}$ is the restriction of $\dfrac{d}{dt}\,F$ to $\mathbb{W}_2^n.$

Example 1. Let $t_0 = 0,\;\; T = 1,\;\; n = 2,$ and

$$F = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Consider the operator $F\dfrac{d}{dt}$ defined on $\mathbb{W}_2^n.$ The Cantor function $t \mapsto k(t)$ is continuous and almost everywhere differentiable on $[0, 1],$ but not absolutely continuous. Therefore, $(0, k(\cdot)) \notin \mathbb{W}_2^n.$ On the other hand, according to the Bernstein theorem, the function $k(\cdot)$ can be approximated uniformly in t by the polynomials

$$B_n(t) := \sum_0^n k\left(\frac{i}{n}\right)\binom{n}{i}\; t^i\,(1-t)^{n-i}.$$

For each $n,$ the polynomial $B_n(\cdot)$ is an absolutely continuous function. Therefore,

$$x_n(\cdot) := \begin{pmatrix} 0 \\ B_n(\cdot) \end{pmatrix} \in \mathbb{W}_2^n.$$

On the other hand, $F\dfrac{d}{dt}\,x_n(\cdot) = (0, 0)$ for any $n,$ and x_n converges to $x(\cdot) := (0, k(\cdot))$ in $\mathbb{L}_2^n,$ which implies that

$$x_n(\cdot) \to x(\cdot) \quad \text{and} \quad F\,\frac{d}{dt}\,x_n(\cdot) \to (0, 0)$$

simultaneously. If the operator $F\dfrac{d}{dt}$ is closed, then $x(\cdot) \in \mathbb{W}_2^n$ by virtue of the closedness of the graph of $F\dfrac{d}{dt}$ [15, p. 17]. However, this inclusion is not true by virtue of the choice of $x(\cdot).$ Consequently, *the operator* $F\,\dfrac{d}{dt}$ *is not closed.*

On the other hand, $x(\cdot) \in \mathbb{W}_2^F.$ Therefore, for the operator $\dfrac{d}{dt}\,F$ defined on $\mathbb{W}_2^F,$ we get

$$x_n(\cdot) \to x(\cdot), \quad \frac{d}{dt}Fx_n(\cdot) \to (0, 0) \;\Rightarrow\; x(\cdot) \in \mathbb{W}_2^F, \quad \frac{d}{dt}\,Fx(\cdot) = (0, 0).$$

4. Conditions for the Normal Solvability of the Operator $\mathcal{D}$

The property of normal solvability of the operator of a system is very important for various applications of linear differential equations. This property enables one to speak of the continuous dependence of a solution with the least norm on the right-hand side. The fact that the operator $\mathcal{D}$ is closed and densely defined enables one to use methods of the theory of ill-posed problems [16, 17] for the investigation of the problem of the normal solvability of $\mathcal{D}$ (Theorems 2 and 3).

The theorem presented below gives a criterion for the existence of a pseudosolution[8] of Eq. (2) for given $(f(\cdot), f_0)$.

Theorem 2. *The boundary-value problem*

$$\frac{d}{dt}Fx(t) = C(t)x(t) + z(t) + f(t),$$

$$\frac{d}{dt}F'z(t) = -C'(t)z(t) + \varepsilon^2 x(t), \tag{4}$$

$$F'z(T) = 0, \quad Fx(t_0) - F'^{+}F'z(t_0) - d = f_0, \quad F'd = 0,$$

has a unique solution $(x(\cdot,\varepsilon), z(\cdot,\varepsilon), d(\varepsilon))$ *for every* $\varepsilon > 0$. *For arbitrary* $(f(\cdot), f_0) \in \mathbb{L}_2^m \times \mathbb{R}^m$, *the descriptor equation*

$$\frac{d}{dt}Fx(t) = C(t)x(t) + f(t), \quad Fx(t_0) = f_0, \tag{5}$$

has a pseudosolution $\widehat{x}(\cdot)$ *if and only if*

$$\|x(\cdot,\varepsilon)\|_2 \le C \quad \text{as} \quad \varepsilon \to 0.$$

Proof. We fix $g(\cdot) := (f(\cdot), f_0) \in \mathbb{L}_2^m \times \mathbb{R}^m$ and show that (4) has a unique solution. We choose $\varepsilon > 0$ and consider the operator $\frac{1}{\varepsilon}\mathcal{D}$. The problem of the projection

$$\left\|\frac{1}{\varepsilon}Dp(\cdot) - g(\cdot)\right\|_1^2 + \|p(\cdot)\|_2^2 \to \min_{p(\cdot)\in\mathscr{D}(\mathcal{D})} \tag{6}$$

of the vector $(0, g(\cdot))$ to the graph of the operator $\frac{1}{\varepsilon}\mathcal{D}$ has a unique solution $\hat{p}(\cdot) \in \mathscr{D}(\mathcal{D})$ by virtue of the closedness of $\frac{1}{\varepsilon}\mathcal{D}$ because the product of the bounded operator of multiplication by a scalar and a closed operator is a closed operator. Since[9]

$$\left\|\frac{1}{\varepsilon}\mathcal{D}p(\cdot) - g(\cdot)\right\|_1^2 + \|p(\cdot)\|_2^2 = \|\mathcal{D}x(\cdot) - g(\cdot)\|_1^2 + \varepsilon^2\|x(\cdot)\|_2^2$$

[8] Here, a pseudosolution is understood as a vector $\hat{x}(\cdot) \in \mathscr{D}(\mathcal{D})$ with the least norm for which $\langle \mathcal{D}\hat{x}(\cdot) - (f(\cdot), f_0), \mathcal{D}x(\cdot)\rangle = 0 \ \ \forall x(\cdot) \in \mathscr{D}(\mathcal{D})$.

[9] The symbol $\|\cdot\|_1$ denotes the norm in the Hilbert space $\mathbb{L}_2^m \times \mathbb{R}^m$.

for $p(\cdot) = \varepsilon x(\cdot),$ by virtue of the arguments presented above the optimization problem

$$\|\mathcal{D}x(\cdot) - g(\cdot)\|_1^2 + \varepsilon^2\|x(\cdot)\|_2^2 \to \min_{x(\cdot)\in\mathscr{D}(\mathcal{D})}$$

has a unique solution $\hat{x}_\varepsilon(\cdot) \in \mathscr{D}(\mathcal{D})$ that satisfies the system of operator equations

$$\mathcal{D}x(\cdot) - z(\cdot) = g(\cdot),$$

$$-\mathcal{D}^* z(\cdot) - \varepsilon^2 x(\cdot) = 0.$$

It is easy to verify that this system is equivalent to (4).

The function

$$\mathscr{L}(\varepsilon) := \min_{x(\cdot)\in\mathscr{D}(\mathcal{D})} \|\mathcal{D}x(\cdot) - g(\cdot)\|_1^2 + \varepsilon^2\|x(\cdot)\|_2^2 = \|\mathcal{D}\hat{x}_\varepsilon(\cdot) - g(\cdot)\|_1^2 + \varepsilon^2\|\hat{x}_\varepsilon(\cdot)\|_2^2$$

satisfies the relation [16, p. 119] (Lemma 1.25)

$$\mathscr{L}(0+) = \mathscr{L}(0) = \min_{x(\cdot)\in\mathscr{D}(\mathcal{D})} \|\mathcal{D}x(\cdot) - g(\cdot)\|_1^2.$$

Thus,

$$\lim_{\varepsilon\to 0} \|\mathcal{D}\hat{x}_\varepsilon(\cdot) - g(\cdot)\|_1^2 = \min_{x(\cdot)\in\mathscr{D}(\mathcal{D})} \|\mathcal{D}x(\cdot) - g(\cdot)\|_1^2. \tag{7}$$

It is known [17, p. 124] (Theorem 18.5) that the convergence of the sequence $\{\hat{x}_\varepsilon(\cdot)\}$ as $\varepsilon \to 0$ is equivalent to the existence of $\hat{x}(\cdot) \in \mathscr{D}(\mathcal{D})$ such that

$$\|\mathcal{D}\hat{x}(\cdot) - g(\cdot)\|_1^2 = \min_{x(\cdot)} \|\mathcal{D}x(\cdot) - g(\cdot)\|_1^2.$$

Let $f(\cdot) \in \operatorname{cl}\mathcal{R}(\mathcal{D})$. Then $\mathcal{D}\hat{x}_\varepsilon(\cdot) \to f(\cdot),$ and the set $\{\mathcal{D}\hat{x}_\varepsilon(\cdot)\}$ is bounded. If $\{\hat{x}_\varepsilon(\cdot)\}$ is also bounded, then, by virtue of the weak closedness of $\mathcal{D},$ we have $f(\cdot) \in \mathcal{R}(\mathcal{D})$. Otherwise, for $f(\cdot) \notin \mathcal{R}(\mathcal{D}),$ the norms of the functions $\{\hat{x}_\varepsilon(\cdot)\}$ increase unboundedly as $\varepsilon \to 0$ by virtue of (7).

In the context of the theorem proved, the following lemma may also be useful:

Lemma 1. *For a rectangular $m \times n$ matrix $F,$ there exist square matrices L and R such that*

$$F = L\Lambda R, \quad F^+ = R'\Lambda^+ L', \quad LL' = E_m, \quad RR' = E_n, \quad \Lambda = \begin{pmatrix} D^{\frac{1}{2}} & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{pmatrix}, \tag{8}$$

where E_m is the identity matrix of order $m,$ $0_{k,s}$ is the zero $k \times s$ matrix, and D is the diagonal matrix of order r formed by the positive eigenvalues $\lambda_1, \dots, \lambda_r$ of the matrix FF'.

Proof. Using the theorem on the singular-value decomposition of a rectangular matrix F (see [18, p. 52]), we write $F = PD^{\frac{1}{2}}Q$, where P and Q are, respectively, $m \times r$ and $r \times n$ rectangular matrices and $P'P = QQ' = E_r$.

We decompose the matrix

$$P = \begin{pmatrix} P_{r,r} \\ P_{m-r,r} \end{pmatrix}$$

into blocks. The columns of the matrix P are orthonormal because $P'P = E_r$. Using the Gram–Schmidt orthogonalization, we can complement them by $m - r$ vectors w_k to the orthonormal basis of the corresponding Euclidean space. We represent the matrix T whose columns are the $m - r$ vectors w_k in the form

$$T = \begin{pmatrix} T_{r,m-r} \\ T_{m-r,m-r} \end{pmatrix}.$$

Then $T'P = 0$ and $T'T = E_{m-r}$ by construction. We set

$$L := \begin{pmatrix} P_{r,r} & T_{r,m-r} \\ P_{m-r,r} & T_{m-r,m-r} \end{pmatrix}.$$

Then $L'L = E_m \Rightarrow LL' = E_m$ because $P'P = E_r$, $T'P = 0$, and $T'T = E_{m-r}$. By analogy, we define

$$R := \begin{pmatrix} Q_{r,r} & Q_{r,n-r} \\ W_{n-r,r} & W_{n-r,n-r} \end{pmatrix},$$

where $Q = \begin{pmatrix} Q_{r,r} & Q_{r,n-r} \end{pmatrix}$ and $W = \begin{pmatrix} W_{n-r,r} & W_{n-r,n-r} \end{pmatrix}$ is a matrix whose rows are $n - r$ vectors v_k that complement the orthonormal row vectors of the matrix Q to the orthonormal basis of the corresponding Euclidean space. Thus, $RR' = E_n$. One can directly verify that $R\Lambda L = PD^{\frac{1}{2}}Q$.

Let us show that $F^+ = R'\Lambda^+L'$. We determine $(L\Lambda R)^+$. It is known [18, p. 69] that $(AB)^+ = B^+A^+$ if and only if $\mathcal{R}(BB'A') \subseteq \mathcal{R}(A')$ and, simultaneously, $\mathcal{R}(A'AB) \subseteq \mathcal{R}(B)$ for rectangular matrices A and B. According to the results proved above, we have $L'L = E_m$. Therefore, $\mathcal{R}(\Lambda RR'\Lambda L') \subset \mathcal{R}(L')$ and $\mathcal{R}(L'L\Lambda R) = \mathcal{R}(\Lambda R)$, whence $(L\Lambda R)^+ = (\Lambda R)^+L'$ because $L^+ = L^{-1} = L'$. Since $RR' = E_n$, we conclude that $(\Lambda R)^+ = R'\Lambda^+$ because $\mathcal{R}(RR'\Lambda') = \mathcal{R}(\Lambda')$ and $\mathcal{R}(\Lambda'\Lambda R) \subset \mathcal{R}(R)$.

The lemma is proved.

In the theorem below, a class of equations of special structure that generate a normally solvable operator is separated from the singular systems (2).

Theorem 3. *Suppose that*

$$L'C(t)R' = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix},$$

where L and R are the same as in Lemma 1 and C_i are stationary rectangular matrices of consistent dimensions. If[10]

$$\sup_{1>\varepsilon>-1} \|Q(\varepsilon)C_2'\|_{\mathrm{mod}} < +\infty, \quad Q(\varepsilon) := (\varepsilon^2 E + C_4' C_4)^{-1},$$

then the operator $\mathcal{D}$ has a closed range of values.

Proof. Let us show that, under the conditions of the theorem, the norms of solutions of the boundary-value problem (4) remain bounded as $\varepsilon \to 0$. To this end, we rewrite (4) in the form

$$\frac{d}{dt}\begin{pmatrix} F & 0 \\ 0 & F' \end{pmatrix}\begin{pmatrix} x(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} C(t) & E \\ \varepsilon^2 & -C'(t) \end{pmatrix}\begin{pmatrix} x(t) \\ z(t) \end{pmatrix} + \begin{pmatrix} f(t) \\ 0 \end{pmatrix}.$$

Using the equality (see Lemma 1)

$$\begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda' \end{pmatrix} = \begin{pmatrix} L' & 0 \\ 0 & R \end{pmatrix}\begin{pmatrix} F & 0 \\ 0 & F' \end{pmatrix}\begin{pmatrix} R' & 0 \\ 0 & L \end{pmatrix},$$

we obtain the equivalent equation

$$\frac{d}{dt}\begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda' \end{pmatrix}\begin{pmatrix} p(t) \\ q(t) \end{pmatrix} = \begin{pmatrix} L'C(t)R' & E \\ \varepsilon^2 & -RC'(t)L \end{pmatrix}\begin{pmatrix} p(t) \\ q(t) \end{pmatrix} + \begin{pmatrix} g(t) \\ 0 \end{pmatrix},$$

where $p(t) := Rx(t)$, $q(t) := L'z(t)$, and $g(t) := L'f(t)$.

The conditions $F'z(T) = 0$, $Fx(t_0) = F'^{+}F'z(t_0) + d + f_0$, and $F'd = 0$ take the form

$$\Lambda' q(T) = 0, \quad \Lambda p(t_0) = \Lambda'^{+}\Lambda' q(t_0) + L'd + L'f_0, \quad \Lambda' L' d = 0.$$

Taking into account the form of Λ, we obtain the system of algebraic–differential equations

$$D^{\frac{1}{2}}\dot{p}_1(t) = C_1 p_1(t) + C_2(t)p_2(t) + g_1(t) + D^{-\frac{1}{2}}v_1(t),$$

$$0 = C_3 p_1(t) + C_4 p_2(t) + g_2(t) + q_2(t),$$

$$\dot{v}_1(t) = \varepsilon^2 p_1(t) - C_1' D^{-\frac{1}{2}} v_1(t) - C_3' q_2(t), \tag{9}$$

$$0 = \varepsilon^2 p_2(t) - C_2' D^{-\frac{1}{2}} v_1(t) - C_4' q_2(t),$$

$$p_1(t_0) - D^{-1}v_1(t_0) = D^{-\frac{1}{2}} g_1^0, \quad d_2 = g_2^0, \quad v_1(T) = 0,$$

[10] For a matrix F, we set $\|F\|_{\mathrm{mod}} := \sum_{i,j} |F_{ij}|$.

where

$$v_1(t) := D^{\frac{1}{2}} q_1(t), \quad p(t) := \begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix}, \quad q(t) := \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix},$$

$$g_0 := L' f_0 = \begin{pmatrix} g_1^0 \\ g_2^0 \end{pmatrix}, \quad L'd = \begin{pmatrix} 0 \\ d_2 \end{pmatrix}.$$

We write the algebraic equations separately:

$$\begin{pmatrix} C_4 & E \\ \varepsilon^2 E & -C_4' \end{pmatrix} \begin{pmatrix} p_2(t) \\ q_2(t) \end{pmatrix} = \begin{pmatrix} -C_3 p_1(t) - g_2(t) \\ C_2' D^{-\frac{1}{2}} v_1(t) \end{pmatrix}. \tag{10}$$

Multiplying (10) from the left by

$$\begin{pmatrix} Q(\varepsilon) C_4' & Q(\varepsilon) \\ E - C_4 Q(\varepsilon) C_4' & -C_4 Q(\varepsilon) \end{pmatrix},$$

we get

$$p_2(t) = Q(\varepsilon)(-C_4' C_3 p_1(t) + C_2' D^{-\frac{1}{2}} v_1(t) - C_4' g_2(t)),$$
$$q_2(t) = (E - C_4 Q(\varepsilon) C_4')(-C_3 p_1(t) - g_2(t)) - C_4 Q(\varepsilon) C_2' D^{-\frac{1}{2}} v_1(t). \tag{11}$$

Substituting the obtained relations into (9), we get the following two-point boundary-value problem for a nonnegative-definite Hamiltonian system with parameter:

$$\dot{p}_1(t) = A(\varepsilon) p_1(t) + R(\varepsilon) v_1(t) + g(t, \varepsilon), \quad p_1(t_0) - D^{-1} v_1(t_0) = g_0,$$
$$\dot{v}_1(t) = -A'(\varepsilon) v_1(t) + S(\varepsilon) p_1(t) + \ell(t, \varepsilon), \quad v_1(T) = 0, \tag{12}$$

where

$$g_0 := D^{-\frac{1}{2}} g_1^0, \quad \ell(t, \varepsilon) = C_3'(E - C_4 Q(\varepsilon) C_4') g_2(t),$$

$$A(\varepsilon) = D^{-\frac{1}{2}}(C_1 - C_2 Q(\varepsilon) C_4' C_3), \quad R(\varepsilon) = D^{-1} + D^{-\frac{1}{2}} C_2 Q(\varepsilon) C_2' D^{-\frac{1}{2}},$$

$$S(\varepsilon) = \varepsilon^2 E + C_3'(E - C_4 Q(\varepsilon) C_4') C_3, \quad g(t, \varepsilon) = D^{-\frac{1}{2}}(g_1(t) - C_2 Q(\varepsilon) C_4' g_2(t)).$$

Using the theorem on "lower" and "upper" solutions [19], we can show that the Riccati equation

$$\dot{K}(t) = A(\varepsilon) K(t) + K(t) A'(\varepsilon) - K(t) S(\varepsilon) K(t) + R(\varepsilon), \quad K(t_0) = D^{-1}, \tag{13}$$

has a unique solution $t \mapsto K(t,\varepsilon)$ defined on $[t_0, T]$ for every $\varepsilon > 0,$ and, furthermore, $K(t,\varepsilon)$ is a nonnegative-definite matrix in its domain of definition. Let $q(\cdot,\varepsilon)$ and $\varphi(\cdot,\varepsilon)$ be determined as solutions of the Cauchy problem

$$\dot\varphi(t) = (A(\varepsilon) - K(t,\varepsilon)S(\varepsilon))\varphi(t) + g(t,\varepsilon) - K(t,\varepsilon)\ell(t,\varepsilon), \quad \varphi(t_0) = g_0, \tag{14}$$

$$\dot q(t) = (-A'(\varepsilon) + S(\varepsilon)K(t,\varepsilon))q(t) + S(\varepsilon)\varphi(t) + \ell(t,\varepsilon), \quad q(T) = 0. \tag{15}$$

Performing the substitution, one can verify that the functions

$$p_1(t,\varepsilon) = K(t,\varepsilon)q(t,\varepsilon) + \varphi(t,\varepsilon), \quad v_1(t,\varepsilon) = q(t,\varepsilon) \tag{16}$$

satisfy (12).

Let us show that, under the conditions of the theorem, the norm of $\varphi(\cdot,\varepsilon)$ remains bounded as $\varepsilon \to 0.$ Indeed,

$$\varphi(t,\varepsilon) = \Phi(t,t_0,\varepsilon)f_1^0 + \int\limits_{t_0}^{t} \Phi(t,s,\varepsilon)(g(s,\varepsilon) - K(s,\varepsilon)\ell(s,\varepsilon))ds,$$

where $\Phi(t,s,\varepsilon)$ is the normalized fundamental matrix for (14) for fixed $\varepsilon > 0.$ It remains to show the boundedness of the functions that are elements of the matrices $\Phi(t,s,\varepsilon)$ and $K(t,\varepsilon)$ and vectors $g(s,\varepsilon)$ and $\ell(s,\varepsilon)$ as $\varepsilon \to 0.$ We set $g(t) := D^{-\frac{1}{2}}(g_1(t) - C_2C_4^+g_2(t))$ and $\ell(t) := C_3'(E - C_4C_4^+)g_2(t).$ Then, as $\varepsilon \to 0,$ we get

$$\int\limits_{t_0}^{T} (g(t,\varepsilon) - g(t), g(t,\varepsilon) - g(t))\,dt$$

$$= \int\limits_{t_0}^{T} (D^{-1}(C_2C_4^+ - C_2Q(\varepsilon)C_4')g_2(t), (C_2C_4^+ - C_2Q(\varepsilon)C_4')g_2(t))dt \to 0$$

because $Q(\varepsilon)C_4' \to C_4^+$ as $\varepsilon \to 0.$ Hence, there exist $\overline{g}, \overline{\ell} > 0$ such that

$$\|g(\cdot,\varepsilon)\|_2 \le \overline{g}, \quad \|\ell(\cdot,\varepsilon)\|_2 \le \overline{\ell}, \quad \varepsilon \to 0.$$

We introduce a function $K_l(\cdot,\varepsilon)$ as a solution of the Bernoulli equation

$$\dot K_l(t) = A(\varepsilon)K_l(t) + K_l(t)A'(\varepsilon) + R(\varepsilon), \quad K_l(t_0) = D^{-1}.$$

Differentiating the function $t \mapsto ((K_l(t,\varepsilon) - K(t,\varepsilon))p, p),$ we obtain

$$(K_l(t,\varepsilon)p, p) \ge (K(t,\varepsilon)p, p), \quad t_0 \le t \le T, \quad \varepsilon > 0,$$

which implies that $\mathrm{sp}\,(K_l(t,\varepsilon) \geq \mathrm{sp}\,(K(t,\varepsilon))$. Let $\|F\|_{\mathrm{sp}} = \mathrm{sp}(FF')$. Then, for a certain $U > 0$, we have

$$\frac{1}{U}\,\|K(t,\varepsilon)\|_{\mathrm{mod}} \leq \|K(t,\varepsilon)\|_{\mathrm{sp}} \leq \mathrm{sp}(K(t,\varepsilon)) \leq \|K_l(t,\varepsilon)\|_{\mathrm{mod}}$$

because $K(t,\varepsilon)$ is a nonnegative-definite symmetric matrix for $t \geq t_0$ and $\varepsilon > 0$. Taking into account that $\|A(\varepsilon) - A\|_{\mathrm{mod}} \to 0$ as $\varepsilon \to 0$ for $A := D^{\frac{1}{2}}(C_1 - C_2C_4^{+}C_3)$ and using the representation

$$K_l(t,\varepsilon) = \exp((A(\varepsilon) + A'(\varepsilon))(t - t_0)) + \int\limits_{t_0}^{t} \exp(A(\varepsilon)(t-s))R(\varepsilon)\exp(A'(\varepsilon)(t-s))ds,$$

one can easily establish that the following relations hold for sufficiently small $\varepsilon > 0$:

$$\|K_l(t,\varepsilon)\|_{\mathrm{mod}} \leq \|\exp((A + A')(t - t_0))\|_{\mathrm{mod}} + 1,$$

$$M\int\limits_{t_0}^{t}(\|\exp(A(t-s))\|_{\mathrm{mod}} + 1)(1 + \|\exp(A'(t-s))\|_{\mathrm{mod}})ds := \overline{K}_1(t),$$

where

$$\|R(\varepsilon)\|_{\mathrm{mod}} \leq M.$$

Thus,

$$\|K(t,\varepsilon)\|_{\mathrm{mod}} \leq U\overline{K}_1(t).$$

Setting

$$P(t,\varepsilon) := A(\varepsilon) - K(t,\varepsilon)S(\varepsilon),$$

we get

$$\|P(t,\varepsilon)\|_{\mathrm{mod}} \leq 1 + \|A\|_{\mathrm{mod}} + U\overline{K}_1(t)(\|C_3'(E - C_4C_4^{+})C_3\|_{\mathrm{mod}} + 1).$$

Using this result, one can easily establish (see, e.g., [12, p. 432]) the boundedness of the elements $\Phi(t,s,\varepsilon)$ as $\varepsilon \to 0$.

By analogy, we prove the boundedness of $q(\cdot,\varepsilon)$. Using equalities (10) and (15), we establish the boundedness of $p_1(\cdot,\varepsilon)$ and $p_2(\cdot,\varepsilon)$ as $\varepsilon \to 0$. Taking into account that $x(t,\varepsilon) = R'p(t,\varepsilon)$ and using Theorem 2, we obtain the required statement.

We illustrate Theorem 2 by an example of a descriptor system of special form that generates an injective operator $\mathcal{D}$ with nonclosed range of values.

Example 2. We set

$$F = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad C(t) \equiv \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

It is easy to verify that, in this case, the operator $\mathcal{D}$ is injective. Hence, the range of values of $\mathcal{D}^*$ is dense in $\mathbb{L}_2^n$ and consists (see Theorem 1) of all vector functions of the form

$$\left\{ \begin{pmatrix} -\dot{z}_1 - z_1 - z_2 \\ z_1 \end{pmatrix},\ z_1 \in \mathbb{W}_2^1([t_0,T]),\ z_1(T) = 0,\ z_2 \in \mathbb{L}_2([t_0,T]) \right\}.$$

This implies that $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \notin \mathcal{R}(\mathcal{D}^*)$ if $f_2 \notin \mathbb{W}_2^1([t_0,T])$. Thus, the range of values of $\mathcal{D}^*$ and, hence, $\mathcal{R}(\mathcal{D})$ are not closed. Note that the conditions of Theorem 3 are not satisfied because $C_2'(\varepsilon^2 E + C_4' C_4)^{-1} = -\varepsilon^{-2}$.

Let us show that, in this case, a solution of Eq. (5) can also be approximated with the use of solutions of (12) for $(f(\cdot), f_0) \in \mathcal{R}(\mathcal{D})$. It is easy to verify that the boundary-value problem (12) takes the form

$$\begin{gathered} \dot{x}_1(t) = x_1(t) + (1+\varepsilon^{-2}) z_1(t) + f_1(t), \\ \dot{z}_1(t) = -z_1(t) + (1+\varepsilon^2) x_1(t) + f_2(t), \\ x_1(t_0) - z_1(t_0) = f_{01}, \quad z_1(T) = 0, \quad x_2(t) = -\varepsilon^2 z_1(t,\varepsilon). \end{gathered} \tag{17}$$

The Riccati equation (13) has the form

$$\dot{k}(t) = 2k(t) + (1+\varepsilon^{-2}) - (1+\varepsilon^2)k^2(t) := U(t,k), \quad k(t_0) = 1. \tag{18}$$

We set

$$k^- := \frac{\varepsilon^2 - \sqrt{\varepsilon^2 + 3\varepsilon^4 + \varepsilon^6}}{\varepsilon^2 + \varepsilon^4}, \quad k^+ := \frac{\varepsilon^2 + \sqrt{\varepsilon^2 + 3\varepsilon^4 + \varepsilon^6}}{\varepsilon^2 + \varepsilon^4}.$$

Using the Picard–Lindelöf theorem, we establish that, for a certain $\varepsilon_0 > 0$, we have

$$k^- < k(t,\varepsilon) < k^+, \quad 0 < \varepsilon < \varepsilon_0, \quad t > t_0, \tag{19}$$

which yields $\dot{k}(t,\varepsilon) > 0$, $t \geq t_0$, $0 < \varepsilon < \varepsilon_0$, because $U(t,k) = (1+\varepsilon^2)(k - k^-)(k^+ - k)$. Thus, $k(t,\varepsilon) \geq k(t_0,\varepsilon) > 0$ for $t \geq t_0$ and $0 < \varepsilon < \varepsilon_0$.

Let $q(\cdot,\varepsilon)$ denote a solution of the equation

$$q_{tt}(t) - 2q_t(t) + (1+\varepsilon^{-2})(1+\varepsilon^2)q(t) = 0, \quad q_t(t_0) = 1 + \varepsilon^2, \quad q(t_0) = 1.$$

By direct substitution, we establish that the function $t \mapsto \dfrac{q_t(t,\varepsilon)}{(1+\varepsilon^2)q(t,\varepsilon)}$ satisfies (18). Hence,

$$q(t,\varepsilon) = e^{\int_{t_0}^t (1+\varepsilon^2)k(s,\varepsilon)ds} > 0 \ \Rightarrow\ q_t(t,\varepsilon) \geq 0, \quad t \geq t_0, \quad 0 < \varepsilon < \varepsilon_0.$$

Carrying out differentiation, one can easily verify that

$$\varphi(t,\varepsilon) = \frac{e^{t-t_0}}{q(t,\varepsilon)} \left\{ f_1^0 + \int\limits_{t_0}^{t} \left(\frac{q(\tau,\varepsilon)}{e^{\tau - t_0}}\, f_1(\tau) - \frac{\dot{q}(\tau,\varepsilon) f_2(\tau)}{e^{\tau - t_0}(1+\varepsilon^2)} \right) d\tau \right\}$$

is a solution of (14) and

$$z(t,\varepsilon) = -\frac{q(t,\varepsilon)}{e^t}\int\limits_t^T \frac{e^s}{q(s,\varepsilon)}\big(f_2(s) + (1+\varepsilon^2)\varphi(s,\varepsilon)\big)\,ds$$

is a solution of (15). Therefore, $x_1(t,\varepsilon) = k(t,\varepsilon)z(t,\varepsilon) + \varphi(t,\varepsilon)$ and $x_2(t,\varepsilon) = -\varepsilon^{-2}z(t,\varepsilon)$.

If, in addition, $f_1(t) \equiv 0$, $f_2(t) = -e^{t-t_0}$, and $f_1^0 = 1$, then the equation $\mathcal{D}x(\cdot) = (f(\cdot), f_0)$ has the unique solution

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \quad x_1(t) = -f_2(t), \quad x_2(t) \equiv 0$$

by virtue of the injectivity of $\mathcal{D}$. On the other hand, we have

$$\varphi(t,\varepsilon) = \frac{\varepsilon^2 e^{t-t_0}}{(1+\varepsilon^2)q(t,\varepsilon)} + \frac{e^{t-t_0}}{1+\varepsilon^2}, \quad z(t,\varepsilon) = -\varepsilon^2\frac{q(t,\varepsilon)}{e^{t+t_0}}\int\limits_t^T \frac{e^{2s}}{q^2(s,\varepsilon)}\,ds.$$

We show that $x_1(\cdot,\varepsilon) \to x_1$ and $x_2(\cdot,\varepsilon) \to 0$ in $\mathbb{L}_2^n$. Since the function $q(\cdot,\varepsilon)$ is increasing, we get

$$\frac{e^{t-t_0}}{1+\varepsilon^2} < \varphi(t,\varepsilon) \le \frac{e^{t-t_0}}{1+\varepsilon^2} + \frac{\varepsilon^2 e^{t-t_0}}{(1+\varepsilon^2)q(t_0,\varepsilon)}, \quad z(t,\varepsilon) \le -\varepsilon^2\frac{q(t,\varepsilon)}{e^{t+t_0}q^2(t,\varepsilon)}\int\limits_t^T e^{2s}ds.$$

Using the equality $q(t_0,\varepsilon) = 1$ and relation (19), we obtain

$$\int\limits_{t_0}^T (\varphi(t,\varepsilon) + f_2(t))^2\,dt \le \int\limits_{t_0}^T \left(\frac{e^{t-t_0}}{1+\varepsilon^2} + \frac{\varepsilon^2 e^{t-t_0}}{(1+\varepsilon^2)} - e^{t-t_0}\right)^2 dt \to 0, \quad \varepsilon \to 0,$$

$$\int\limits_{t_0}^T k^2(t,\varepsilon)z^2(t,\varepsilon)\,dt \le \int\limits_{t_0}^T \left(-\varepsilon^2 k^+ \frac{e^{2T} - e^{2t}}{2e^{t+t_0}}\right)^2 dt \to 0, \quad \varepsilon \to 0.$$

Thus, $x_1(\cdot,\varepsilon) \to -f_2(\cdot)$.

One can verify that

$$q(t,\varepsilon) = \frac{(\varepsilon^4 + \sqrt{\varepsilon^2 + 3\varepsilon^4 + \varepsilon^6})e^{\frac{\varepsilon^2 + \sqrt{\varepsilon^2+3\varepsilon^4+\varepsilon^6}}{\varepsilon^2}(t-t_0)}}{2\sqrt{\varepsilon^2 + 3\varepsilon^4 + \varepsilon^6}} + \frac{(\sqrt{\varepsilon^2 + 3\varepsilon^4 + \varepsilon^6} - \varepsilon^4)e^{\frac{\varepsilon^2 - \sqrt{\varepsilon^2+3\varepsilon^4+\varepsilon^6}}{\varepsilon^2}(t-t_0)}}{2\sqrt{\varepsilon^2 + 3\varepsilon^4 + \varepsilon^6}}.$$

Therefore, the norm of $q(\cdot,\varepsilon)$ unboundedly increases as $\varepsilon \to 0$. On the other hand, we have

$$-\varepsilon^{-2}z(t,\varepsilon) \le \frac{e^{2T} - e^{2t}}{2e^{t+t_0}q(t,\varepsilon)}.$$

Therefore, $x_2(\cdot,\varepsilon) \to 0$.

The next example illustrates the application of sufficient conditions for normal solvability (Theorem 3) to a descriptor equation that is not decomposed into algebraic and differential components.

Example 3. We set

$$F = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad C(t) \equiv \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

The corresponding descriptor system has the form

$$\frac{d}{dt}\left(\frac{1}{2}\,x_1 - \frac{1}{2}\,x_2\right)(t) = -\frac{1}{2}\,x_1(t) - \frac{1}{2}\,x_2(t) + f_1(t),$$

$$\frac{d}{dt}\left(-\frac{1}{2}\,x_1 + \frac{1}{2}\,x_2\right)(t) = \frac{1}{2}x_1(t) + \frac{1}{2}\,x_2(t) + f_2(t),$$

$$\left(\frac{1}{2}\,x_1 - \frac{1}{2}\,x_2\right)(t_0) = f_1^0, \left(-\frac{1}{2}x_1 + \frac{1}{2}\,x_2\right)(t_0) = f_2^0.$$

We set

$$T := \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \Lambda := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then

$$F = T\Lambda T' \quad \text{and} \quad T'C(t)T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

According to Theorem 3, the corresponding operator $\mathcal{D}$ is normally solvable because $C_2Q(e) \equiv 0$. On the other hand, the closedness of the range of values of the operator

$$p(\cdot) \mapsto \mathcal{D}_1 p(\cdot) = \left(\frac{d}{dt}\,\Lambda p(\cdot) - T'C(t)Tp(\cdot),\ \Lambda\, p(t_0)\right)$$

can be verified directly. Indeed, the adjoint operator acts according to the rule

$$(z(\cdot), z_0) \mapsto \begin{pmatrix} -\dot{z}_1(t) - z_2(t) \\ 0 \end{pmatrix}, \quad z_2 \in \mathbb{L}_2(t_0, T), \quad z_1 \in \mathbb{W}_2^1(t_0, T), \quad z_1(T) = 0.$$

Therefore, $\mathcal{R}(\mathcal{D}_1^*) = \mathbb{L}_2(t_0, T) \times \{0\}$ is a closed set together with $\mathcal{R}(\mathcal{D}_1)$.

Note that $\det(\lambda F + C) \equiv 0$. The kernel of the operator $\mathcal{D}$ is infinite-dimensional because

$$(F - C(t))T \begin{pmatrix} 0 \\ f(\cdot) \end{pmatrix} = 0, \quad f(\cdot) \in \mathbb{L}_2(t_0, T).$$

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