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STATE ESTIMATION FOR A DYNAMICAL SYSTEM DESCRIBED BY A LINEAR EQUATION WITH UNKNOWN PARAMETERS

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We investigate the state estimation problem for a dynamical system described by a linear operator equation with unknown parameters in a Hilbert space. In the case of quadratic restrictions on the unknown parameters, we propose formulas for *a priori* mean-square minimax estimators and *a posteriori* linear minimax estimators. A criterion for the finiteness of the minimax error is formulated. As an example, the main results are applied to a system of linear algebraic–differential equations with constant coefficients.

Introduction

One of the main problems in contemporary applied mathematics is the state estimation problem for a dynamical system described by linear equations with unknown parameters. This problem belongs to the broad class of problems known as *inverse problems under conditions of indeterminacy.* Mathematically, this class of problems can be described as follows: On the basis of a given element (observations of a state, output measurements, etc.) of a certain functional space, find an estimator for an element $l(\theta)$ under the condition that θ satisfies the relation $g(\theta) = 0$. Problems of the determination of $l(\theta)$ are informative if the equation $g(\theta) = 0$ has the set of solutions and $y = C(\theta)$ for a certain element θ of this set. Thus, the estimation problem can be formulated in this case as follows: On the basis of a given $y = C(\theta)$, $\theta \in \Theta$, $y \in Y$, find an estimator $\widehat{l(\theta)}$ for the element $l(\theta)$ under the condition that $g(\theta) = 0$ and $C(\cdot)$ and $l(\cdot)$ are known functions. Note that, in the case where the equation $y = C(\theta)$ has a unique solution $\hat{\theta}$, the estimation problem degenerates in the sense that the expression $l(\hat{\theta})$ is the unique estimator for $l(\theta)$.

We call an estimation problem linear if Θ and Y are linear spaces and $C(\cdot)$ and $l(\theta)$ are linear mappings. One of the classes often considered is the class of linear problems defined by the functions

$$C(\theta) = H\varphi + D\eta, \qquad g(\theta) = L\varphi + Bf, \qquad \theta = (x, f, \eta) \subset X \times F \times Y,$$

where H, D, L, and B are linear operators. We call a linear estimation problem an *estimation problem under conditions of indeterminacy* if $D \neq 0$, L or B is not equal to zero, and if $B = 0$, then $N(L) = \{\varphi : L\varphi = 0\} \neq \{0\}$. Note that the *type of indeterminacy* determines the choice of a method for the solution of the estimation problem: if f and η are realizations of random elements, then it is natural to use the stochastic approach. In this case, one needs *a priori* information on characteristics of the distribution of random elements. Assume that indeterminacy takes place if the distribution of random elements is partially unknown or some deterministic parameters are partially unknown. For details of various statements of estimation problems under conditions of indeterminacy (jointly known as the *theory of guaranteed estimation*) for different l, L, H, B, and D and for special spaces, see, e.g., [1]. For the classical results of the theory of guaranteed estimation, see [2–4].

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In the classical theory, the assumption of the existence of the bounded inverse of the operator of a system is essential. The linear estimation problem under conditions of indeterminacy for equations with noninjective operator in an abstract Hilbert space was studied, in particular, in [5], where estimators were found in the case of quadratic restrictions on the unknown parameters. The developed method essentially uses the finite dimensionality of the kernel and cokernel of the operator of the system and its normal solvability. Therefore, estimators can be written, in particular, for boundary-value problems for systems of normal linear ordinary differential equations. In [6], a criterion was proposed for the solvability of Noetherian boundary-value problems for linear algebraic–differential equations with varying coefficients (the term "descriptor systems" is also widely used in the literature) under the condition that algebraic–differential equations can be reduced to the central canonical form [7, p. 57]. In particular, this condition guarantees the unique solvability of the corresponding Cauchy problem [7, p. 67]. Combining these results with results of [5], one can construct estimators for solutions of Noetherian boundary-value problems for linear descriptor equations of special structure with unknown parameters. On the other hand, an example of a linear descriptor equation with constant coefficients for which the homogeneous Cauchy problem has only the trivial solution whereas the operator induced by the Cauchy problem has the nonclosed set of values was given in [8]. The estimation methods proposed in [1–3, 5] cannot be directly applied to these systems.

The main result of the present paper is a method for the guaranteed estimation of equations with a linear closed densely defined operator in an abstract Hilbert space. The main advantage of this method is that it does not require the Noetherian property of an operator system and its normal solvability. This method is the development of our approach proposed in [9, 10] for linear algebraic–differential equations in spaces of square summable vector functions; it generalizes the results of [1–3] to the case of linear equations with unbounded operator. For Noetherian equations, the obtained representations of estimators [11] coincide with those described in [5]. As an example, we apply the proposed method to the state estimation problem for a linear algebraic–differential equation with constant coefficients. Moreover, the reduction to the central canonical form is not required.

We now introduce the necessary notation: $c(G,\cdot) = \sup\{(z,f), f \in G\}$ is the support function of the set G, $\delta(\mathcal{G},\cdot)$ is the indicator function of $\mathcal{G}$, $\operatorname{dom} f = \{x \in \mathcal{H} : f(x) < \infty\}$ is the effective set of a function f, $f^*(x^*) = \sup_x \{(x^*, x) - f(x)\}$ is the Young–Fenchel transformation or the function conjugate to f, $\operatorname{cl} f = f^{**}$ is the closure of the function f, for eigenfunctions the function f coincides with the lower semicontinuous regularization of f, $(fL)(x) = f(Lx)$ is the image of the function f under the linear operator L, $(L^*c)(u) = \inf\{c(G,z), L^*z = u\}$ is the preimage of the function $c(G,\cdot)$ under the operator L^*, $\operatorname{Arginf}_u f(u)$ is the collection of the points of minimum of the function f, P_{L^*} is the operator of orthogonal projection onto $R(L^*)$, $\partial f(x)$ is the subdifferential of the function f at the point x, and $(\cdot,\cdot)$ is the scalar product of a Hilbert space.

Statement of the Problem

Assume that an operator φ satisfies the condition $L\varphi \in \mathcal{G}$, and a vector y is defined and associated with φ by the relation

$$y = H\varphi + \eta. \tag{1}$$

The operators L and H and the set $\mathcal{G}$ are assumed to be given, and the element η "simulates" indeterminacy (e.g., it is a random vector). Our aim is to solve the following inverse problem: On the basis of the given y, construct the operation of estimation $\widehat{l(\varphi)}$ of the expression $l(\varphi)$ and determine the estimation error σ. Let us formulate this more rigorously.

Let L be a closed operator that maps an everywhere dense subset $\mathfrak{D}(L)$ of a Hilbert space $\mathcal{H}$ into a Hilbert space $\mathcal{F}$ and let $H \in \mathcal{L}(\mathcal{H}, \mathcal{Y})$. We represent the condition $L\varphi \in \mathcal{G}$ in the following equivalent form: Assume that φ satisfies the linear operator equation

$$L\varphi = f, \tag{2}$$

where the right-hand side f is a certain unknown element $\mathcal{G} \subset \mathcal{F}$. Thus, we know that one of solutions φ of Eq. (2) for a certain $f \in \mathcal{G}$ is defined by the given vector y up to the element η and the operator H: $H\varphi = y - \eta$. In what follows, we assume that the element η simulates two types of indeterminacy: it denotes a realization of a random vector with values in $\mathcal{Y}$, mean value zero, and a correlation operator $R_\eta \in \mathfrak{R}$, where $\mathfrak{R}$ is a given subset of $\mathcal{L}(\mathcal{Y}, \mathcal{Y})$, η is a deterministic vector, $(f, \eta) \in G$, and G is a given subset of $\mathcal{F} \times \mathcal{Y}$.

Note that the realization of y is determined not only by specific η, H, and f. In the general case, $N(L) = \{\varphi \in \mathfrak{D}(L) : L\varphi = 0\}$ is a nontrivial linear manifold. Therefore, $y = H(\varphi_0 + \varphi) + \eta$, where φ_0 is an arbitrary element of the unbounded set $N(L)$.

We set $l(\varphi) = (l, \varphi)$. We seek the estimator $\widehat{l(\varphi)}$ in the class of affine functionals $\widehat{l(\varphi)} = (u, y) + c$ of observations. We do not assume here that the operators L and H have bounded inverse operators. Therefore, small deviations on the right-hand side of (2) and in measurements (1) can lead to an infinitely large estimation error. Taking into account this remark and the series of indeterminacies indicated above, we construct an operation of estimation on the basis of the minimax approach. This gives a guaranteed estimation error that characterizes the maximum deviation of the estimator from the actual value and is finite for a fairly broad collection of pairs of operators L and H.

Note that we can consider two statements of the estimation problem: *a posteriori* statement and *a priori* statement. In the case of *a priori* estimation, in the process of the construction of an operation of estimation we rely on the "worst" realization y, analyzing all possible correlation operators R_η and the right-hand sides f. As a result, the optimal estimator is determined only by the direction l and the structure of given sets of restrictions.

Definition 1. *The affine functional* $\widehat{\widehat{l(\varphi)}} = (\hat{u}, \cdot) + \hat{c}$ *determined from the condition*

$$\sigma(l, \hat{u}) = \inf_{u,c} \sigma(l, u), \qquad \sigma(l, u) := \sup_{L\varphi \in \mathcal{G}, R_\eta \in \mathfrak{R}} M\big(l(\varphi) - \widehat{\widehat{l(\varphi)}}\big)^2$$

is called the a priori mean-square minimax estimator of the expression $l(\varphi) = (l, \varphi)$. *The number* $\hat{\sigma}(l) = \sigma^{1/2}(l, \hat{u})$ *is called the mean-square minimax error in the direction* l.

An *a posteriori* operation of estimation associates a specific realization of y with the "Chebyshev center" of the set $\mathcal{X}_y \subset \mathcal{H}$ (so-called *a posteriori* set) of all possible φ each of which is consistent with "measured" y by virtue of (1) and (2):

$$(L\varphi,\ y - H\varphi) \in G.$$

Therefore, we seek this estimator only among the elements of $\mathcal{X}_y$. Note that the inclusion $(L\varphi, y - H\varphi) \in G$ implies the inequality $\|y\| < C$, where the constant C is defined by the structure of G. Therefore, in the case of *a posteriori* estimation, there is no reason to assume that the undefined element η is a random process because the inequality $\|R_\eta\| < c$ for the norm of the correlation operator does not guarantee that $\|y\| < C$ for a specific realization η. For this reason, we assume that the undefined element η is deterministic.

Definition 2. *The set*

$$\mathcal{X}_y = \{\varphi \in \mathcal{D}(L) : (L\varphi, y - H\varphi) \in \mathcal{G}\}$$

is called the a posteriori set, the vector $\hat{\varphi}$ *is called the a posteriori minimax estimator for a vector* φ *in a direction* l *if*

$$\hat{d}(l) := \inf_{\varphi \in \mathcal{X}_y} \sup_{\psi \in \mathcal{X}_y} |(l, \varphi) - (l, \psi)| = \sup_{\psi \in \mathcal{X}_y} |(l, \hat{\varphi}) - (l, \psi)|,$$

and the expression $\hat{d}(l)$ *is called the a posteriori minimax error in the direction* l.

Main Results

Below, we describe the general form of an *a priori* mean-square minimax estimator and formulate a criterion for the finiteness of the error of the mean-square minimax estimator.

Proposition 1. *Let* $\mathcal{G}$ *and* $\mathcal{R}$ *be convex closed bounded subsets of* $\mathcal{F}$ *and* $\mathcal{L}(\mathcal{Y}, \mathcal{Y})$, *respectively. For given* $l \in \mathcal{H}$, *the minimax error* $\hat{\sigma}(l)$ *is finite if and only if, for a certain* $u \in \mathcal{Y}$, *one has*

$$l - H^* u \in \operatorname{dom} \operatorname{cl}(L^* c) \cap (-1) \operatorname{dom} \operatorname{cl}(L^* c).$$

For these u *and* l, *one has*

$$\sigma(l, u) = \frac{1}{4}\left[\operatorname{cl}(L^* c)(l - H^* u) + \operatorname{cl}(L^* c)(-l + H^* u)\right]^2 + \sup_{R_\eta \in \mathcal{R}} (R_\eta u, u), \tag{3}$$

and, furthermore,

$$R(L^*) \subset \operatorname{dom} \operatorname{cl}(L^* c) \subset \overline{R(L^*)}.$$

If $\operatorname{Arginf}_u \sigma(l, u) \neq \varnothing$, *then* $\widehat{\widehat{l(\varphi)}} = (\hat{u}, y) + \hat{c}$, *where*

$$\hat{u} \in \operatorname{Arginf}_u \sigma(l, u), \qquad \hat{c} = \frac{1}{2}\left(\operatorname{cl}(L^* c)(l - H^* \hat{u}) - \operatorname{cl}(L^* c)(-l + H^* \hat{u})\right).$$

Theorem 1. *Suppose that* $\mathcal{G}$ *is a convex closed bounded symmetric set whose interior contains* 0, *and the random element* η *satisfies the condition*

$$\eta \in \{\eta : M(\eta, \eta) \leq 1\}.$$

Then, for given $l \in \mathcal{H}$, *the minimax error* $\hat{\sigma}(l)$ *is finite if and only if* $l - H^*u \in R(L^*)$ *for a certain* $u \in \mathcal{Y}$. *For these* l, *there exists a unique mean-square minimax estimator* $\hat{u} \in \mathcal{U}_l$, *which is determined from the condition*

$$\sigma(l, \hat{u}) = \min_u \sigma(l, u),$$
$$\sigma(l, u) = (u, u) + \min_z \{c^2(\mathcal{G}, z),\ L^*z = l - H^*u\}. \tag{4}$$

If the sets $R(L)$ *and* $H(N(L))$ *are closed, then* $\hat{u}$ *is determined from the condition*

$$\hat{u} - Hp_0 \in H\big(\partial I_2(H^*\hat{u})\big), \qquad Lp_0 = 0,$$
$$I_2(w) = \min_z \{c^2(\mathcal{G}, z),\ L^*z = P_{L^*}(l - w)\}. \tag{5}$$

Corollary 1. *Suppose that*

$$\mathcal{G} = \{f \in \mathcal{F} : (f, f) \le 1\}, \qquad \eta \in \{\eta : M(\eta, \eta) \le 1\}$$

and one of the following conditions is satisfied:

(i) the sets $R(L)$ *and* $H(N(L))$ *are closed;*

(ii) the set $R(T) = \{[Lx, Hx],\ x \in \mathcal{D}(L)\}$ *is closed.*

Then, for $l \in R(L^*) + R(H^*)$, *and only for these* l, *the unique minimax estimator* $\hat{u}$ *can be represented in the form* $\hat{u} = H\hat{p}$, *where* $\hat{p}$ *is an arbitrary solution of the system*

$$L^*\hat{z} = l - H^*H\hat{p},$$
$$L\hat{p} = \hat{z}. \tag{6}$$

The mean-square minimax error has the form

$$\hat{\sigma}(l) = (l, \hat{p})^{1/2}.$$

Corollary 2. *Suppose that linear operators* $L : \mathcal{H} \mapsto \mathcal{F}$ *and* $H \in \mathcal{L}(\mathcal{H}, \mathcal{Y})$ *satisfy condition (i) or (ii) of Corollary 1. Then the system of operator equations (6) has a solution* $\hat{z} \in \mathcal{D}(L^*)$, $\hat{p} \in \mathcal{D}(L)$ *if and only if* $l = L^*z + H^*u$ *for certain* $z \in \mathcal{D}(L^*)$ *and* $u \in \mathcal{Y}$.

Corollary 3. *Under the conditions of Corollary 1, for an arbitrary* $l \in R(L^*) + R(H^*)$ *and a realization of* $y(\cdot)$, *the representation* $(\hat{u}, y) = (l, \hat{\varphi})$ *is true, where* $\hat{\varphi}$ *is determined from the system*

$$L^*\hat{q} = H^*(y - H\hat{\varphi}),$$
$$L\hat{\varphi} = \hat{q}. \tag{7}$$

We now consider *a posteriori* estimators.

Proposition 2. *Let* $\mathcal{G}$ *be a convex closed bounded subset of* $\mathcal{Y} \times \mathcal{F}$. *An a posteriori minimax error in the direction* l *is finite if and only if* $l \in \operatorname{dom} c(\mathcal{X}_y, \cdot) \bigcap (-1)\operatorname{dom} c(\mathcal{X}_y, \cdot)$ *and*

$$R(L^*) + R(H^*) \subset \operatorname{dom} c(\mathcal{X}_y, \cdot) \bigcap (-1)\operatorname{dom} c(\mathcal{X}_y, \cdot) \subset \overline{R(L^*) + R(H^*)}. \tag{8}$$

For these l, *the estimator and the error are as follows:*

$$(l, \hat{\varphi}) = \frac{1}{2}\Big(c(\mathcal{X}_y, l) - c(\mathcal{X}_y, -l)\Big), \qquad \hat{d}(l) = \frac{1}{2}\Big(c(\mathcal{X}_y, l) + c(\mathcal{X}_y, -l)\Big).$$

Theorem 2. *Suppose that*

$$\mathcal{G} = \Big\{(f, \eta) : \|f\|^2 + \|\eta\|^2 \le 1\Big\}$$

and the operators L *and* H *satisfy condition (i) or (ii) of Corollary 1. Then, for* $l \in R(L^*) + R(H^*)\mathcal{G}_1$, *and only for these* l, *an a posteriori minimax estimator* $\hat{\varphi}$ *for a vector* φ *in the direction* l *exists and is determined from system (7). The a posteriori error is determined by the relation*

$$\hat{d}(l) = \big(1 - (y, y - H\hat{\varphi})\big)^{1/2}\hat{\sigma}(l). \tag{9}$$

Corollary 4. *Suppose that, under the conditions of Theorem 2, for an arbitrary direction* l *one has* $\widehat{l(\varphi)} = (l, \hat{\varphi})$, *where* $\hat{\varphi}$ *is determined from (7). Then the vector* $\hat{\varphi}$ *is a minimax estimator for the vector* φ *in the sense that*

$$\inf_{\varphi \in \mathcal{X}_y} \sup_{x \in \mathcal{X}_y} \|\varphi - x\| = \sup_{x \in \mathcal{X}_y} \|\hat{\varphi} - x\| = \big(1 - (y, y - H\hat{\varphi})\big)^{1/2} \max_{\|l\|=1} \hat{\sigma}(l).$$

Let us illustrate the application of Corollary 4. Without loss of generality (see the lemma on singular decomposition in [8]), we can assume that the matrices F and C are determined by a collection of blocks of consistent dimensions, i.e.,

$$F = \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}.$$

Proposition 3. *Suppose that* $t \mapsto x(t) \in \mathbb{R}^n$ *is determined as a solution of the equation*

$$\frac{d}{dt}Fx(t) - Cx(t) = f(t), \qquad Fx(t_0) = 0,$$

and the set $\mathcal{G}$ *has the form*

$$\mathcal{G} = \left\{(f, \eta) : \int_{t_0}^{T} \left(\|f(t)\|^2 + \|\eta(t)\|^2\right) dt \le 1\right\}.$$

Then the a posteriori minimax estimator for the function $x(\cdot)$ *based on the observations* $y(t) = x(t) + \eta(t)$, $t_0 \le t \le T$, *is defined by the expression* $\hat{x}(\cdot)$, *where* $\hat{x}(t) = [x_1(t), x_2(t)]$ *and the functions* $x_1(\cdot)$ *and* $x_2(\cdot)$ *are determined from the equations*

$$\begin{aligned}
\dot{x}_1(t) &= (C_1 - C_2(E + C_4'C_4)^{-1}C_4'C_3)x_1(t) + (C_2(E + C_4'C_4)^{-1}C_2' + E)q_1(t) \\
&\quad + C_2(E + C_4'C_4)^{-1}y_2(t), \qquad x_1(t_0) = 0, \\
\dot{q}_1(t) &= (-C_1' + C_3'C_4(E + C_4'C_4)^{-1}C_2')q_1(t) + C_3'C_4(E + C_4'C_4)^{-1}y_2(t) - y_1(t) \\
&\quad + (C_3'(E - C_4(E + C_4'C_4)^{-1}C_4')C_3 + E)x_1(t), \qquad q_1(T) = 0, \\
x_2(t) &= -(E + C_4'C_4)^{-1}C_4'C_3x_1(t) + (E + C_4'C_4)^{-1}(C_2'q_1(t) + y_2(t)), \\
q_2(t) &= -(E - C_4(E + C_4'C_4)^{-1}C_4')C_3x_1(t) - C_4(E + C_4'C_4)^{-1}(C_2'q_1(t) + y_2(t)).
\end{aligned} \tag{10}$$

The minimax error has the form

$$\sup_{\mathcal{X}_y}\|x - \hat{x}\| = \left(1 - \int_{t_0}^{T}(y, y - \hat{x})dt\right)^{1/2} \max_{\|l\|=1}\left(\int_{t_0}^{T}(l, p)dt\right)^{1/2},$$

where the function $p(\cdot)$ *is determined from (10) if one sets* $y(t) = l(t)$.

The proposition remains true for a nonstationary matrix $C(t)$.

Auxiliary Results and Proof

We introduce the sets

$$\mathcal{U}_l = \left\{u \in Y : L^*z = l - H^*u\right\}, \qquad D = \left\{l \in \mathcal{H} : \mathcal{U}_l \ne \varnothing\right\},$$

where, upon the identification of the Hilbert spaces $\mathcal{H}$ and $\mathcal{F}$ with their duals, the operator L^* acts from $\mathcal{F}$ into $\mathcal{H}$. The unique existence of the adjoint operator L^* is guaranteed by the fact that L is densely defined [12, p. 40]. Recall that the indicator function $\delta(\mathcal{G}, \cdot)$ of the set $\mathcal{G}$ is defined as follows: $\delta(\mathcal{G}, f) = 0$ for $f \in \mathcal{G}$, and $\delta(\mathcal{G}, f) = +\infty$ for $f \notin \mathcal{G}$.

The lemma below plays the key role in the proof of the theorem on the existence, uniqueness, and representation of minimax estimators.

Lemma 1. *Let $\mathcal{G}$ be a convex bounded closed subset of $\mathcal{F}$ and let L be a linear densely defined closed operator from $\mathcal{H}$ into $\mathcal{F}$. Then*

$$(L^*c)^* = (\delta L), \qquad (L^*c)^{**} = (\delta L)^*, \qquad R(L^*) \subset \operatorname{dom}(\delta L)^* \subset \overline{R(L^*)}.$$

If the interior of $\mathcal{G}$ has common points with $R(L)$, then $\operatorname{dom}(\delta L)^* = \operatorname{dom}(L^*c) = R(L^*)$, the functional (L^*c) is an eigenfunctional, $L^*c = (L^*c)^{**}$, and

$$(L^*c)(x) = c(\mathcal{G}, z_0) = \inf\{c(\mathcal{G}, z) | L^*z = x\}, \qquad x \in R(L^*).$$

The lemma remains true if the indicator function of a convex set is replaced by a convex eigenfunction [9].

Remark 1. The condition $\operatorname{int}\mathcal{G} \bigcap R(L) \neq \varnothing$ of Lemma 1 is essential because there exist an operator L and a set $\mathcal{G}$ such that $R(L) \neq \overline{R(L)}$, $\operatorname{int}\mathcal{G} = \varnothing$, $\operatorname{dom}(L^*c) = R(L)$, $\operatorname{dom}(\delta L)^* = \overline{R(L)}$, and $(L^*c)(x) > (\delta L)^*(x)$ for $x \in \overline{R(L)} / R(L)$. Indeed, we set

$$F = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad C(t) \equiv \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

and define an operator $x \mapsto Lx \in (\mathbb{L}_2[0,1])^2$ by the method described in the proof of Proposition 3. The equation $Lx = 0$ is equivalent to the system of algebraic–differential equations

$$\dot{x}_1(t) - x_1(t) + x_2(t) = 0, \qquad x_1(0) = 0, \qquad -x_1(t) = 0,$$

which implies that $x_{1,2}(t) = 0$ on $[0, 1]$. Therefore, $N(L) = \{0\}$ because $\overline{R(L^*)} = (\mathbb{L}_2[0,1])^2$. On the other hand, for the solvability of the algebraic–differential equation

$$-\dot{z}_1(t) - z_1(t) - z_2(t) = f_1(t), \qquad z_1(1) = 0, \qquad z_1(t) = f_2(t),$$

it is necessary that $f_2(\cdot)$ be absolutely continuous. Therefore, $R(L^*)$ and $R(L)$ are not closed. We set

$$\mathcal{G} = \left\{ f = (f_1, f_2) : \int_0^1 f_1^2(s)ds \le 1,\ f_2 = 0 \right\}.$$

Then $\operatorname{int}\mathcal{G} = \varnothing$ in $(\mathbb{L}_2[0,1])^2$. Since $Lp \in \mathcal{G} \Leftrightarrow p_1 = 0$, $\|p_2\| \le 1$, we have

$$(\delta L)^*(x) = \sup\{(x, p) - \delta(\mathcal{G}, Lp)\} = \sup\left\{ (p_2, x_2),\ \int_0^1 p_2^2(s)ds \le 1 \right\} = \|x_2\|.$$

Thus,

$$\operatorname{dom}(\delta L)^* = \overline{R(L^*)} = (\mathbb{L}_2[0,1])^2.$$

On the other hand,

$$c(\mathcal{G}, z) = c(P\mathcal{S}_1(0), z) = c(\mathcal{S}_1(0), P^*z) = \|z_1\|,$$

$$S_1(0) = \left\{f \in \mathbb{L}_2^2[0,1] : \|f\| \le 1\right\},$$

where P denotes the operator of multiplication by the matrix $\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$ in the space $\mathbb{L}_2^2[0,1]$. By virtue of the injectiveness of L^*, we get

$$(L^*c)(x) = \inf\left\{c(\mathcal{G}, z) : L^*z = x\right\} = \|x_2\|, \qquad x_{1,2} \in \mathbb{W}_2^1[0,1].$$

If

$$x_n = (x_1, x_{2,n}) \to x = (x_1, x^*), \qquad x^* \notin \mathbb{W}_2^1[0,1],$$

then

$$(L^*c)(x_n) \to \left\|x^*\right\| = (\delta L)^*(x),$$

but $(L^*c)(x) = +\infty$.

Proof of Lemma 1. Let $p \in \mathcal{D}(L)$. Since $p \in \mathcal{D}(L)$, the linear functional $z \mapsto p(z) = (p, L^*z)$ is bounded. Therefore, it can be extended to the entire space $\mathcal{F}$ by continuity. Hence,

$$(L^*c)^*(p) = \sup_{x \in R(L^*)}\left\{(p, x) - \inf\left\{c(\mathcal{G}, z) \mid L^*z = x\right\}\right\}$$

$$= \sup_{x \in R(L^*)}\sup_{z \in L^{*-1}(x)}\left\{(p, x) - c(\mathcal{G}, z)\right\} = \sup_{z \in \mathcal{D}(L^*)}\left\{(p, L^*z) - c(\mathcal{G}, z)\right\}$$

$$= \sup_{z \in \mathcal{F}}\left\{(Lp, z) - c(\mathcal{G}, z)\right\} = c^*(\mathcal{G}, \cdot)(Lp) = \delta(\mathcal{G}, Lp).$$

Consider the case where $p \notin \mathcal{D}(L)$. By the definition of the operator adjoint to a bounded linear operator [12, p. 39], the linear functional $z \mapsto p(z) = (p, L^*z)$ is unbounded. This means that there exists a sequence $\{z_n\}$ such that $\|z_n\| \le 1$, $z_n \in \mathcal{D}(L^*)$, and $p(z_n) \to +\infty$. On the other hand, the support function $c(\mathcal{G}, \cdot)$ of a bounded convex set is bounded in the neighborhood of an arbitrary point $z \in \mathcal{F}$, and, hence, it is continuous [13, p. 21]. Then $\sup_n c(\mathcal{G}, z_n) = M < +\infty$ and

$$(L^*c)^*(p) = \sup_{z \in \mathcal{D}(L^*)} \{(p, L^*z) - c(\mathcal{G}, z)\} \geq \sup_n \{p(z_n) - M\} = +\infty.$$

On the other hand, by definition, we have $(\delta L)(p) = +\infty$. Thus, we have shown that $(L^*c)^*(p) = (\delta L)(p)$ for all p, whence $(L^*c)^{**} = (\delta L)^*$.

Let $x \notin N(L)^{\perp}$ and let $Lp \in \mathcal{G}$ for a certain $p \in \mathcal{D}(L)$. There exists $p_0 \in N(L)$ such that $n(p_0, x) > 0$, $n \in \mathbb{N}$. In this case, we have

$$(\delta L)^*(x) = \sup_{q \in \mathcal{D}(L)} \{(q, x) - \delta(\mathcal{G}, Lq)\} \geq \sup_{n \in \mathbb{N}} \{n(p_0, x)\} = +\infty.$$

Therefore,

$$\operatorname{dom}(\delta L)^* \subset N(L)^{\perp} = \overline{R(L^*)}.$$

On the other hand, if $x = L^*z$, then

$$(\delta L)^*(x) = \sup_{q \in \mathcal{D}(L)} \{(Lq, z) - \delta(\mathcal{G}, Lq)\} \leq \sup_{f \in \mathcal{F}} \{(f, z) - \delta(\mathcal{G}, f)\} = c(\mathcal{G}, x) < +\infty$$

because $\mathcal{G}$ is bounded. Therefore,

$$R(L^*) \subset \operatorname{dom}(\delta L)^* \subset \overline{R(L^*)}.$$

Now assume that $\operatorname{int}\mathcal{G} \cap R(L) \neq \varnothing$. Let us show that this is sufficient for the validity of the inequality

$$(L^*c) \leq (\delta L)^*.$$

Indeed,

$$x^* \in \operatorname{dom}(\delta L)^*, \quad x \in \mathcal{D}(L) \Rightarrow (x^*, x) - (\delta L)^*(x^*) \leq \delta L(x) < +\infty$$

by virtue of the Young–Fenchel inequality [14]. We fix $x^* \in \operatorname{dom}(\delta L)^*$ and introduce the set

$$\mathcal{M}(x^*) = \{(z, \mu) \mid Lx = z,\ \mu = (x^*, x) - (\delta L)^*(x^*)\}.$$

Note that

$$\mathcal{W} := \operatorname{int}\operatorname{epi}(\delta(\mathcal{G}, \cdot)) = \operatorname{int}\mathcal{G} \times \{\mu \in \mathbb{R}^1 : \mu > 0\} \cap \mathcal{M}(x^*) = \varnothing.$$

Indeed, if $(z, \mu) \in \mathcal{W} \cap \mathcal{M}$, then

$$\delta(\mathcal{G}, Lx) < \mu = (x^*, x) - (\delta L)^*(x^*), \qquad Lx = z,$$

which contradicts the Young–Fenchel inequality.

Thus, the convex sets $\operatorname{epi}(\delta(\mathcal{G}, \cdot))$ and $\mathcal{M}(x^*)$ can be separated by a nonzero linear continuous functional (z^0, β_0):

$$\sup\left\{(z^0, z) + \beta_0\alpha \,\middle|\, (z, \alpha) \in \mathcal{W}\right\} \le \inf\left\{(z^0, z) + \beta_0\alpha \,\middle|\, (z, \alpha) \in \mathcal{M}(x^*)\right\}. \tag{11}$$

It is easy to verify that $\beta_0 < 0$. Indeed, if $\beta_0 > 0$, then the supremum in (11) is equal to $+\infty$. On the other hand, the supremum in (11) is never equal to $-\infty$, which guarantees that the infimum in (11) is finite. If $\beta_0 = 0$, then, according to (11), $\mathcal{G}$ and $R(L)$ are separated by the functional $(z^0, \cdot)$, but then $\operatorname{int}\mathcal{G} \cap R(L) = \varnothing$.

By the definition of $\mathcal{M}(x^*)$, we have

$$-\infty < (\mathcal{G}, z^0) = \sup\left\{(z^0, z) - \beta_0\delta(\mathcal{G}, z)\right\} \le \inf\left\{(z^0, Lx) - |\beta_0|(x^*, x)\right\} + |\beta_0|(\delta L)^*(x^*),$$

whence

$$-\infty < \inf_x\left\{(z^0, Lx) - \beta_0(x^*, x)\right\} \quad \Rightarrow \quad \left[-|\beta_0|x^*, z^0\right] \perp \{[x, Lx],\ x \in \mathcal{D}(L)\}.$$

Taking into account the form of the orthogonal complement of the graph of L [12, p. 40], we obtain

$$z_0 \in \mathcal{D}(L^*), \qquad L^* z_0 = |\beta_0|x^* \quad \Rightarrow \quad (L^*c)(x^*) \le c(\mathcal{G}, \beta_0^{-1}z^0) \le (\delta L)^*(x^*).$$

We have shown that, on $\operatorname{dom}(\delta L)^*$, one has $(L^*c) = (\delta L)^*$ and $\operatorname{dom}(\delta L)^* \subset R(L^*)$. By definition, $R(L^*) \subset \operatorname{dom}(L^*c)$. We have proved earlier that $R(L^*) \subset \operatorname{dom}(\delta L)^*$. Generally speaking, we have $(L^*c) \ge (L^*c)^{**} = (\delta L)^*$. Therefore, $\operatorname{dom}(\delta L)^* \subset \operatorname{dom}(L^*c)$. Thus,

$$(L^*c) = (\delta L)^*, \qquad \operatorname{dom}(\delta L)^* = \operatorname{dom}(L^*c) = R(L^*).$$

According to the Fenchel–Moreau theorem, $(L^*c) = (L^*c)^{**} = (\delta L)^*$ if and only if (L^*c) has a closed supergraph, which, for convex eigenfunctionals, is equivalent to the lower semicontinuity [14, p. 178].

The lemma is proved.

Proof of Proposition 1. Taking the equality $M\xi^2 = M(\xi - M\xi)^2 + (M\xi)^2$ and relation (1) into account, we get

$$M\left((l, \varphi) - (u, y) - c\right)^2 = \left[(l - H^*u, \varphi) - c\right]^2 + M(u, \eta)^2.$$

Hence,

$$\sup_{\varphi \in L^{-1}(\mathcal{G}),\, R_\eta \in \mathcal{R}} M\big((l, \varphi) - (u, y) - c\big)^2 = \sup_{\varphi \in L^{-1}(\mathcal{G})} \big[(l - H^* u, \varphi) - c\big]^2 + \sup_{R_\eta \in \mathcal{R}} (R_\eta u, u).$$

We transform the first term as follows:

$$\sup_{\varphi \in L^{-1}(\mathcal{G})} \big[(l - H^* u, \varphi) - c\big]$$

$$= \frac{1}{2}\Big((\delta L)^*(l - H^* u) + (\delta L)^*(-l + H^* u)\Big) + \left|c - \frac{1}{2}\Big((\delta L)^*(l - H^* u) - (\delta L)^*(-l + H^* u)\Big)\right|. \tag{12}$$

Using relation (12), for given l, u, and c we get

$$\sup_{\varphi \in L^{-1}(\mathcal{G})} \big[(l - H^* u, \varphi) - c\big] < +\infty \quad \Leftrightarrow \quad l - H^* u \in \operatorname{dom}(\delta L)^* \cap -\operatorname{dom}(\delta L)^*.$$

The set $\operatorname{dom}(\delta L)^*$ is a convex cone with vertex at zero. Therefore, $\operatorname{dom}(\delta L)^* \cap -\operatorname{dom}(\delta L)^*$ is the maximum linear manifold contained in $\operatorname{dom}(\delta L)^*$. Setting

$$c = \frac{1}{2}\Big((\delta L)^*(l - H^* u) - (\delta L)^*(-l + H^* u)\Big)$$

and using relation (12) and Lemma 1, we obtain the expression for $\sigma(l, u)$.

The expression $\sup_{R_\eta \in \mathcal{R}} (R_\eta u, u)$ is finite for an arbitrary u. Indeed,

$$(R_\eta u, u) \le \|R_\eta\| \|u\|^2 \le \|u\|^2, \qquad R_\eta \in \mathcal{R}.$$

Therefore, $\sigma(l, u) < +\infty$. To complete the proof, it remains to use the definition of mean-square minimax estimator.

Proof of Theorem 1. According to Proposition 1, for given $l \in \mathcal{H}$ a minimax estimator is finite if and only if

$$l - H^* u \in \operatorname{dom}(\delta L)^* \cap -\operatorname{dom}(\delta L)^*.$$

Since $0 \in \mathcal{G} \cap R(L)$, the conditions of Lemma 1 are satisfied. Therefore, $\operatorname{dom}(\delta L)^* = R(L^*)$ and

$$I_1^{1/2}(u) := \operatorname{cl}(L^* c)(l - H^* u) = (L^* c)(l - H^* u).$$

Using Proposition 1, we establish the statement of the theorem concerning the finiteness of the minimax error. We get

$$(R_\eta u, u) = M(\eta, u)^2 \le M(\eta, \eta)(u, u) \quad \Rightarrow \quad \sup_{R_\eta \in \mathcal{R}} (R_\eta u, u) = (u, u).$$

Using (3), we obtain

$$\sigma(l, u) = I_1(u) + (u, u),$$

and, hence, relation (4) is true. Note that $U_l = \{u : l - H^* u \in R(L^*)\}$ by virtue of the definition of U_l. The functional I_1 is convex and weakly lower semicontinuous, which follows from Lemma 1. Hence, $u \mapsto \sigma(l, u)$ is weakly semicontinuous, strictly convex, and coercive. Since $I_1(u) = +\infty$ in the complement of U_l, for an arbitrary minimizing sequence $\{u_n\}$, we have $u_n \in \mathcal{U}_l$. This sequence is bounded by virtue of the coercivity of $u \mapsto \sigma(l, u)$. We separate a weakly convergent subsequence $\{u_n\}$. By virtue of weak semicontinuity, the greatest lower bound of $u \mapsto \sigma(l, u)$ is attained at the weak limit of the sequence $\{u_n\}$. Thus, the set of points of minimum is nonempty, and, by virtue of strict convexity, it consists of a single point $\hat{u}$. Since $I_1(u) = +\infty$ for $u \notin U_l$, we have $l - H^* \hat{u} \in R(L^*)$. Thus, we have proved the existence and uniqueness of a minimax estimator.

Assume that the condition of the second part of the theorem is satisfied. Then

$$U_l = \{u : P_{N(L)} H^* u = P_{N(L)} l\},$$

where $P_{N(L)}$ denotes the orthoprojector to $N(L)$. Consider the functional

$$I_2(w) = \min_z \{c^2(\mathcal{G}, z),\ L^* z = P_{L^*}(l - w)\}.$$

According to Lemma 1, $I_2(\cdot)$ attains its minimum $\hat{z}(w)$ at every point w. Then, by virtue of properties of a support function, we have

$$I_2^{1/2}(w) = c(\mathcal{G}, \hat{z}(w)) \le c(\mathcal{G}, z(w)) + c^2(\mathcal{G}, z_0),$$

where $L^* z_0 = 0$ and $L^* z(w) = P_{L^*}(l - w)$, $z(w) \in R(L)$. Since the left-hand side of this inequality does not depend on z_0, we get

$$I_2^{1/2}(w) \le c(\mathcal{G}, z(w)) + \min_{z_0 \in N(L^*)} c(\mathcal{G}, z_0) = c(\mathcal{G}, z(w))$$

because $c(\mathcal{G}, \cdot) \ge 0$ and $c(\mathcal{G}, 0) = 0$. For an arbitrary w, the boundedness of $I_2(\cdot)$ in a certain neighborhood $V(w)$ now follows from the statement that $z(w)$ depends continuously on w (L is normally solvable) and from properties of the set $\mathcal{G}$. Thus, $I_2(\cdot)$ is a continuous function. According to the theorem on the subdifferential of the image of a convex function under a linear operator [14, p. 212], we get

$$\partial I_3(\hat{u}) = H \partial I_2(H^* \hat{u}), \qquad I_3(u) = I_2(H^* u).$$

On the other hand, on U_l we have

$$P_{L^*}(l - H^*u) = l - H^*u \quad \Rightarrow \quad I_1(u) = I_2(H^*u) = I_3(u).$$

Therefore, the point of minimum $\hat{u}$ of the functional $\sigma(l, \cdot)$ is simultaneously a solution of the problem of conditional optimization

$$I_4(u) = (u, u) + I_3(u) \quad \rightarrow \quad \min, \quad u \in U_l.$$

Since the affine manifold $\mathcal{U}_l$ is parallel to the linear subspace $\mathcal{U}_0 = \left\{u : P_{N(L)}H^*u = 0\right\}$, a necessary and sufficient condition for an extremum of I_4 on U_l has the form [14, p. 89]

$$\partial I_4(\hat{u}) \cap (\mathcal{U}_0)^{\perp} \neq \varnothing.$$

According to the Moreau–Rockafellar theorem, we have $\partial I_4(\hat{u}) = \partial I_3(\hat{u}) + \{2\hat{u}\}$. On the other hand, by virtue of the conditions of the theorem, we get

$$(\mathcal{U}_0)^{\perp} = N^{\perp}(P_{N(L)}H^*) = \overline{R(P_{N(L)}H^*)^*} = H(N(L)).$$

Thus, there exists $p_0 : Lp_0 = 0$ such that

$$\hat{u} - Hp_0 \in H\partial I_2(H^*\hat{u}),$$

$$I_2(w) = \min_z \left\{c^2(\mathcal{G}, z),\ L^*z = P_{L^*}(l - w)\right\}, \qquad \hat{u} \in \mathcal{U}_l.$$

The theorem is proved.

Proof of Corollary 1. Note that the sets $\mathcal{G}$ and $\{\eta : M(\eta, \eta) \leq 1\}$ satisfy the conditions of Theorem 1. Therefore, there exists a unique mean-square minimax estimator $\hat{u} \in \mathcal{U}_l$. Assume that condition (i) is satisfied. Then, according to Theorem 1, we get

$$\hat{u} - \hat{H}p \in H(\partial I_2(\hat{u})), \quad \hat{u} \in U_l, \quad I_2(w) = \min_z \left\{c^2(\mathcal{G}, z),\ L^*z = P_{L^*}(l - w)\right\}.$$

Let us determine the subdifferential I_2. We introduce additional notation. Let $\tilde{L}_1^*$ denote the linear operator defined on $R(L^*)$ according to the rule

$$\tilde{L}_1^*w = z, \qquad z \in R(L) \cap \mathcal{D}(L^*), \qquad L^*z = w.$$

Let us show that $\tilde{L}_1^*$ is a closed operator. Indeed, let

$$w_n \rightarrow w, \qquad w_n \in R(L^*), \qquad \tilde{L}_1^*w_n = z_n \rightarrow z.$$

Then $w \in R(L^*)$ and $L^* z_n = w_n \to w$, $z_n \to z$. Since $z_n \in R(L)$ by the definition of $\tilde{L}_1^*$, we have $z \in R(L)$. Taking into account that L^* is closed, we establish that $z \in R(L) \cap \mathcal{D}(L^*)$ and $L^* z = w$, i.e., $w \in \mathcal{D}(\tilde{L}_1^*)$ and $\tilde{L}_1^* w = z$.

By virtue of the closed-graph theorem [12], $\tilde{L}_1^*$ is bounded. We extend $\tilde{L}_1^*$ to the entire $\mathcal{F}$ so that

$$L_1^* w = \tilde{L}_1^* (I - P_{N(L)}) w, \qquad w \in \mathcal{F}.$$

We associate the operator L_1 with an operator L by analogy with the construction of L_1^* and establish that $(L_1)^* = L_1^*$. Indeed, for arbitrary $p \in \mathcal{F}$ and $w \in \mathcal{H}$, we have

$$(L_1^* w, p) + (L_1 p, w) = (z, p) + (q, w) = (z, Lq) + (q, L^* z) = 0,$$

where $z \in R(L) \cap \mathcal{D}(L^*)$, $L^* z = w$, $q \in R(L^*) \cap \mathcal{D}(L)$, and $Lq = p$.

Note that $c(\mathcal{G}, z) = \|z\|$. Therefore, for $l - w \in R(L^*)$, by the definition of L_1^* we get

$$I_2^{1/2}(w) = \left\| L_1^* (l - w) \right\| = \min_z \left\{ c(\mathcal{G}, z),\ L^* z = P_{L^*}(l - w) \right\}.$$

Setting

$$k(q) = \left\| L_1^* l - q \right\|^2,$$

we obtain

$$I_2(w) = \left\| L_1^* (l - w) \right\|^2 = k(L_1^* w) = (k L_1^*)(w).$$

Note that $q \mapsto k(q)$ is a convex continuous functional on the entire space $\mathcal{F}$. Therefore, it satisfies the conditions of the theorem on the subdifferential of the image of a convex functional under a linear continuous operator [14, p. 212], whence

$$\partial(k L_1^*)(w) = L_1 \partial k(L_1^* w) = L_1 \partial \left\| L_1^* (l - w) \right\|^2.$$

We set $w = H^* \hat{u}$. Since $\hat{u} \in \mathcal{U}_l$, we have $L^* \hat{z} = l - H^* \hat{u}$, where $\hat{z} = L_1^* (l - H^* \hat{u})$. Thus,

$$\partial I_2(H^* \hat{u}) = \partial(k L_1^*)(H^* \hat{u}) = L_1 \partial \|\hat{z}\|^2 = 2\|\hat{z}\| L_1(\overline{\mathcal{G}}_1(\hat{z})),$$

where $\overline{\mathcal{G}}_1(z) = \{ f \in \mathcal{G} : (f, z) = \|z\| \}$.

If $\hat{z} = 0$, then

$$0 = L_1^* (l - H^* \hat{u}) = \tilde{L}_1^* (l - H^* \hat{u}).$$

By virtue of the injectivity of $\tilde{L}_1^*$, we get $l = H^*\hat{u}$. Condition (5) takes the form $Hp_0 = \hat{u}$, $Lp_0 = 0$, and, hence, $0 = l - H^*Hp_0$ and $Lp_0 = 0$. Thus, $\hat{u}$ is expressed in terms of solutions of (6).

Let $\hat{z} \neq 0$. By the definition of the operator L_1, we have

$$HL_1(\overline{\mathcal{G}}_1(\hat{z})) = \left\{Hp,\ Lp = \frac{\hat{z}}{\|\hat{z}\|},\ p \in R(L^*)\right\}.$$

Thus, condition (5) takes the form

$$\begin{aligned} \hat{u} - Hp_0 &= 2\|\hat{z}\|Hp, \\ L^*\hat{z} &= l - H^*\hat{u}, \qquad \hat{z} \in R(L), \\ Lp &= \frac{\hat{z}}{\|\hat{z}\|}, \qquad p \in R(L^*), \end{aligned} \tag{13}$$

for a certain $p_0 \in N(L)$. We set $\tilde{p} = 2\|\hat{z}\|^{-1}p$. Using (13), we get $\hat{u} = H(\tilde{p} + p_0)$, where

$$L(\tilde{p} + p_0) = \hat{z}, \qquad Lp_0 = 0,$$

$$L^*\hat{z} = l - H^*H(\tilde{p} + p_0).$$

If we now set $\hat{p} = \tilde{p} + p_0$ for $\tilde{p}$ and p_0, then $\hat{p}$ and $\hat{z}$ satisfy (6). Consequently, $\hat{u} = H\hat{p}$.

We now show that $\hat{p}$ can be taken as an arbitrary solution of (6). Indeed, we introduce the linear operator $Tx = [Lx, Hx]$ from $\mathcal{H}$ into the Cartesian product $\mathcal{F} \times \mathcal{Y}$. It is clear that $N(T) = N(L) \cap N(H)$ and $T^*(u, z) = L^*z + H^*u$. Let (p_0, z_0) be determined from the conditions

$$\begin{aligned} Lp_0 &= z_0, \\ L^*z_0 + H^*Hp_0 &= 0. \end{aligned} \tag{14}$$

We set $u_0 = Hp_0$. Then $T^*(u_0, z_0) = 0$ and $Tp_0 = [z_0, u_0]$. Therefore, $Tp_0 \in N(T^*)$. However,

$$R(T) \cap N(T^*) = \{0\},$$

whence $p_0 \in N(T) = N(L) \cap N(H)$, i.e., $u_0 = 0$. It remains to note that any two solutions (p_1, z_1) and (p_2, z_2) of the linear equation (6) differ by a solution of (14). Therefore, according to the result proved above, we have $H(p_1 - p_2) = 0$.

Now assume that condition (ii) is satisfied. Then the unique solution $[u^*, z^*]$ of the problem of optimization

$$\|[z, u]\|^2 \to \inf, \qquad T^*[z, u] = l \tag{15}$$

is orthogonal to the null manifold T^*, and, hence, it belongs to the range of values of the operator T, i.e., we simultaneously have

$$[u^*, z^*] = Tx, \qquad T^*[u^*, z^*] = l.$$

By the definition of T, this yields

$$Lx = z^*, \qquad Hx = u^*, \qquad L^*z^* + H^*u^* = l.$$

This, in turn, yields

$$u^* \in \mathcal{U}_l \quad \Rightarrow \quad \sigma(\hat{u}, l) \le \sigma(u^*, l).$$

On the other hand, $l = L^*\hat{z} + H^*\hat{u}$ and $Lp = \hat{z}$ by virtue of relation (6) for a certain $p \in \mathfrak{D}(L)$. Thus, $T^*[\hat{u}, \hat{z}] = l$. Therefore, by virtue of (15), we get

$$\sigma(l, \hat{u}) = \|[\hat{u}, \hat{z}]\|^2 \ge \|[u^*, z^*]\|^2.$$

On the other hand, according to (15), we have

$$\sigma(u^*, l) = (u^*, u^*) + \min_z \left\{\|z\|^2,\ L^*z = l - H^*u^*\right\} \le (u^*, u^*) + (z^*, z^*) \le \sigma(\hat{u}, l).$$

Therefore, $\sigma(l, \hat{u}) = \sigma(l, u^*)$, which, by virtue of strict convexity, yields $u^* = \hat{u}$.

Taking (6) into account, we conclude that $\sigma(l, u) = (\hat{z}, \hat{z}) + (\hat{u}, \hat{u}) = (l, p)$, whence $\hat{\sigma}(l) = (l, \hat{p})^{1/2}$.

Corollary 1 is proved.

Proof of Corollary 2. If the system of operator equations (6) has a solution $\hat{z} \in \mathfrak{D}(L^*)$, $\hat{p} \in \mathfrak{D}(L)$, then $l = L^*z + H^*u$ for $\hat{z}$ and $\hat{u}$, $\hat{u} = H\hat{p}$.

Now assume that the conditions of the corollary are satisfied and $l \in R(L^*) + R(H^*)$. Then the operators L and H and the vector l satisfy the conditions of Corollary 1. Therefore, the minimax estimator $\hat{u}$ can be represented in the form $\hat{u} = H\hat{p}$, where $\hat{p}$ is determined as a solution of (6).

Corollary 2 is proved.

Proof of Corollary 3. First of all, note that system (7) has the nonempty set of solutions (q, φ). This follows from Corollary 2 and the statement according to which, for an arbitrary $y \in \mathcal{Y}$, the vector H^*y belongs to the set $R(L^*) + R(H^*)$. Now let $\hat{u} = H\hat{p}$, where $\hat{p}$ is determined as a solution of (6) and $\hat{\varphi}$ is determined as a solution of (7). By direct calculation, one can easily establish that $(\hat{u}, y) = (l, \hat{\varphi})$.

Corollary 3 is proved.

Proof of Proposition 2. We write

$$-c(\mathcal{X}_y, -l) \le (l, \psi) \le c(\mathcal{X}_y, l), \qquad \psi \in \mathcal{X}_y,$$

whence

$$|(l, \psi)| \le \frac{1}{2}\Big(c(\mathcal{X}_y, l) + c(\mathcal{X}_y, -l)\Big), \qquad \psi \in \mathcal{X}_y.$$

Therefore,

$$\sup_{\psi \in \mathcal{X}_y} |(l, \varphi) - (l, \psi)| = \frac{1}{2}\Big(c(\mathcal{X}_y, l) + c(\mathcal{X}_y, -l)\Big) + \Big|(l, \varphi) - \frac{1}{2}\Big(c(\mathcal{X}_y, l) - c(\mathcal{X}_y, -l)\Big)\Big|. \tag{16}$$

Expression (16) is meaningful only if $l \in \operatorname{dom} c(\mathcal{X}_y, \cdot) \bigcap (-1)\operatorname{dom} c(\mathcal{X}_y, \cdot)$. We show that

$$R(L^*) + R(H^*) \subset \operatorname{dom} c(\mathcal{X}_y, \cdot) \subset \overline{R(L^*) + R(H^*)}.$$

The first inclusion is a corollary of the statement that, for an arbitrary $l = L^* z + H^* u$, one has

$$c(\mathcal{X}_y, l) = \sup_{x \in \mathcal{X}_y} \{(Lx, z) - (u, y - Hx)\} + (u, y) \le c(\mathcal{G}, [z, u]) + (u, y) < +\infty$$

by virtue of the boundedness of $\mathcal{G}$.

On the other hand, we have

$$c(\mathcal{X}_y, l) \ge \sup\{(l, x),\ Lx = 0,\ Hx = 0\} = +\infty$$

for every $l \notin \overline{R(L^*) + R(H^*)}$. Thus, expression (16) is meaningful only if condition (8) is satisfied. In what follows, we assume that this condition is satisfied. It follows from (16) that

$$\sup_{\psi \in \mathcal{X}_y} |(l, \varphi) - (l, \psi)| \ge \frac{1}{2}\Big(c(\mathcal{X}_y, l) + c(\mathcal{X}_y, -l)\Big)$$

for any $\varphi \in \mathcal{X}_y$, and the equality is realized for

$$(l, \hat{\varphi}) = \frac{1}{2}\Big(c(\mathcal{X}_y, l) - c(\mathcal{X}_y, -l)\Big), \qquad \hat{\varphi} \in \mathcal{X}_y$$

by virtue of the convexity of $\mathcal{G}$ and the continuity of the scalar product.

Proposition 2 is proved.

Proof of Theorem 2. Assume that the operators L and H satisfy the conditions of the theorem. Then the projection problem

$$\mathcal{I}(x) = (Lx, Lx) + (y - Hx, y - Hx) \quad \to \quad \min_{x \in \mathcal{D}(L)} \tag{17}$$

has a solution $\hat{\varphi}$. Indeed [15, p. 23], for an arbitrary $y \in \mathcal{Y}$, the set of solutions of (17) is, at the same time, a collection of solutions of the variational equality

$$-(L\varphi, Lx) + (y - H\varphi, Hx) = 0, \qquad x \in \mathcal{D}(L), \tag{18}$$

which contains, in particular, the $\hat{\varphi}$-solution of the consistent system (see Corollary 2)

$$L^*\hat{q} = H^*(y - H\hat{\varphi}),$$

$$L\hat{\varphi} = \hat{q}.$$

We set

$$\mathcal{X}_0 = \{x : \mathcal{I}_1(x) + \mathcal{I}(\hat{\varphi}) \le 1\}, \qquad \mathcal{I}_1(x) = (Lx, Lx) + (Hx, Hx).$$

Note that

$$\mathcal{I}(\hat{\varphi} - x) = \mathcal{I}_1(x) + \mathcal{I}(\hat{\varphi}) - 2(L\hat{\varphi}, Lx) + 2(y - H\hat{\varphi}, Hx) = \mathcal{I}_1(x) + \mathcal{I}(\hat{\varphi})$$

for $x \in \mathcal{D}(L)$ by virtue of equality (18).

Let $x \in \mathcal{X}_0$. Then $\mathcal{I}(\hat{\varphi} - x) = \mathcal{I}_1(x) + \mathcal{I}(\hat{\varphi}) \le 1$ and, hence,

$$\hat{\varphi} + (-1)\mathcal{X}_0 = \hat{\varphi} + \mathcal{X}_0 \subset \mathcal{X}_y.$$

Conversely, if $x \in \mathcal{X}_y$, then $\tilde{x} := \hat{\varphi} - x \in \mathcal{D}(L)$ and

$$1 \ge \mathcal{I}(x) = \mathcal{I}(\hat{\varphi} - \tilde{x}) = \mathcal{I}_1(\hat{\varphi} - x) + \mathcal{I}(\hat{\varphi}).$$

Therefore,

$$-x + \hat{\varphi} \in \mathcal{X}_0, \quad x \in \mathcal{X}_y \quad \Rightarrow \quad \mathcal{X}_y \subset \hat{\varphi} + \mathcal{X}_0.$$

Thus,

$$c(\mathcal{X}_y, l) = (l, \hat{\varphi}) + c(\mathcal{X}_0, l).$$

Note that

$$c(\mathcal{X}_0, l) = \sup_x \{(l, x) - \delta(S^0_\beta, Tx)\} = \inf\{c(S^0_\beta, [z, u]) \mid L^*z + H^*u = l\}, \tag{19}$$

where $Tx = [Lx, Hx]$, $\delta(S^0_\beta, \cdot)$ is the indicator function of the ball

$$S^0_\beta = \{[p, q] : s(p, q) \le \beta\}, \qquad s(p, q) = (p, p) + (q, q), \qquad \beta = 1 - \mathcal{I}(\hat{\varphi}) \ge 0.$$

Indeed, by definition, we have

$$x \in \mathcal{X}_0 \quad \Leftrightarrow \quad s(Tx) \le \beta \quad \Leftrightarrow \quad \delta(S^0_\beta, Tx) \le 0.$$

Thus, the convex functional $x \mapsto \delta T(x) = \delta(S^0_\beta, Tx)$ is the indicator function of $\mathcal{X}_0$. Since the linear operator T and the set S^0_β satisfy the conditions of Lemma 1, we get

$$c(\mathcal{X}_0, \cdot) = (\delta T)^*(\cdot) = T^* c(S^0_\beta, \cdot),$$

where $c(S^0_\beta, w) = (w, w)^{1/2}\beta^{1/2}$ by virtue of the Schwarz inequality. Thus, according to (19), we have

$$c(\mathcal{X}_y, l) = (l, \hat{\varphi}) + \beta^{1/2}\Big[\inf\Big\{\|z\|^2 + \|u\|^2,\ L^* z + H^* u = l\Big\}\Big]^{1/2}$$

for an arbitrary $l \in \mathcal{H}$. However, $\inf\{\cdot\}$ on the right-hand side of the last equality is nothing but an *a priori* minimax error [see the arguments presented in the solution of the problem of optimization (15)]. Therefore,

$$c(\mathcal{X}_y, l) = (l, \hat{\varphi}) + \beta^{1/2}\hat{\sigma}(l).$$

To complete the proof it remains to note that [see relation (9)]

$$\beta = 1 - \mathcal{I}(\hat{\varphi}) = 1 - (y, y - H\hat{\varphi})$$

and to use Proposition 2.

Proof of Corollary 4. According to Theorem 2, we get

$$\inf_{\varphi \in \mathcal{X}_y} \sup_{x \in \mathcal{X}_y} \|\varphi - x\| = \inf_{\varphi \in \mathcal{X}_y} \sup_{x \in \mathcal{X}_y} \sup_{\|l\|=1} |(l, \hat{\varphi} - x)|$$

$$\ge \sup_{\|l\|=1} \inf_{\varphi \in \mathcal{X}_y} \sup_{x \in \mathcal{X}_y} \|\varphi - x\| = \sup_{\|l\|=1} \hat{d}(l) = \big(1 - (y, y - H\hat{\varphi})\big)^{1/2} \max_{\|l\|=1} \hat{\sigma}(l).$$

It is clear that

$$\sup_{x \in \mathcal{X}_y} \|\hat{\varphi} - x\| \ge \inf_{\varphi \in \mathcal{X}_y} \sup_{x \in \mathcal{X}_y} \|\varphi - x\|.$$

By virtue of the conditions of the corollary, this yields

$$\sup_{x \in \mathcal{X}_y} \|\hat{\varphi} - x\| = \sup_{\|l\|=1} \sup_{x \in \mathcal{X}_y} |(l, \hat{\varphi} - x)| = \sup_{\|l\|=1} \hat{d}(l) \ge \inf_{\varphi \in \mathcal{X}_y} \sup_{x \in \mathcal{X}_y} \|\varphi - x\|.$$

Corollary 4 is proved.

Proof of Proposition 3. Let $\mathcal{D}$ denote the operator generated by the linear descriptor equation with matrices F and C. In this case, the operator H acts as follows: $Hx = x$. Then the operators $\mathcal{D}$ and H and the set $\mathcal{G}$ satisfy condition (ii). In this case, according to Theorem 2, the *a posteriori* minimax estimator $\hat{x}$ for the solution x of the descriptor equation in the direction l exists for any $l \in R(L^*) + R(H^*) = \mathbb{L}_2^n(t_0, T)$ and is determined from the operator equations

$$
\begin{aligned}
L^*\hat{q} &= H^*(y - H\hat{x}),\\
L\hat{x} &= \hat{q}.
\end{aligned}
\tag{20}
$$

The *a posteriori* error is determined by the relation

$$\hat{d}(l) = \big(1 - (y, y - H\hat{x})\big)^{1/2}(l, \hat{p})^{1/2}.$$

Note that Eqs. (20) are equivalent to the system of algebraic–differential equations

$$
\begin{aligned}
\dot{x}_1(t) &= C_1x_1(t) + C_2x_2(t) + q_1(t), \qquad x_1(t_0) = 0,\\
0 &= C_3x_1(t) + C_4x_2(t) + q_2(t),\\
\dot{q}_1(t) &= -C_1'q_1(t) - C_3'q_2(t) + x_1(t) - y_1(t), \qquad q_1(T) = 0,\\
0 &= -C_2'q_1(t) - C_4'q_2(t) + x_2(t) - y_2(t).
\end{aligned}
\tag{21}
$$

Indeed, taking into account the block structure of the matrices F and C, we can write $x = (x_1, x_2)$ and $z = (z_1, z_2)$. Then

$$Fx(t) = \begin{pmatrix} x_1(t) \\ 0 \end{pmatrix}, \qquad F'z(t) = \begin{pmatrix} z_1(t) \\ 0 \end{pmatrix}, \qquad Cx(t) = \begin{pmatrix} C_1x_1(t) + C_2x_2(t) \\ C_3x_1(t) + C_4x_2(t) \end{pmatrix}.$$

Taking into account the form of L and L^*, we establish the equivalence of (20) and (21).

We rewrite the algebraic equations (21) in the form

$$\begin{pmatrix} C_4 & E \\ E & -C_4' \end{pmatrix}\begin{pmatrix} x_2(t) \\ q_2(t) \end{pmatrix} = \begin{pmatrix} -C_3x_1(t) \\ C_2'q_1(t) + y_2(t) \end{pmatrix}.$$

Multiplying this equality from the left by

$$\begin{pmatrix} (E + C_4'C_4)^{-1}C_4' & (E + C_4'C_4)^{-1} \\ E - C_4(E + C_4'C_4)^{-1}C_4' & -C_4(E + C_4'C_4)^{-1} \end{pmatrix},$$

we obtain the representations of x_2 and q_2 given in the proposition. The expressions for x_1 and q_1 are established by substituting the obtained relations into (21).

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