Infinite horizon optimal control and stabilizability for linear descriptor systems

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Abstract

In this paper we construct an infinite horizon LQ optimal controller for a linear stationary differential-algebraic equation (DAE) and prove that stabilizability from the given initial state is necessary and sufficient for the controller existence. We also solve the problem of finite horizon LQ optimal control. Our approach is based on ideas from linear geometric control theory that allows one to represent solutions of DAEs as outputs of LTI system. It is applicable for generic DAEs without imposing additional regularity assumptions.

1 Introduction

Consider a linear Differential-Algebraic Equation (DAE) in the following form:

$$\frac{d(Ex)}{dt} = Ax(t) + Bu(t), \quad Ex(t_0) = Ex_0 \,, \qquad (1)$$

where $E, A \in \mathbb{R}^{c\times n}$ and $B \in \mathbb{R}^{c\times m}$ are given matrices. In this paper, we are interested in the following two problems. For the given positive definite matrices Q, R and positive semi-definite matrix Q_0, find a control law $u^*(t)$ such that:

(1) **Finite horizon optimal control:** u^* minimizes the cost functional

$$J(u, x_0, t_1) = \inf_{x \text{ satisfies } (1)} (x^T(t_1)TE^TQ_0Ex(t_1) + \int_0^{t_1} x^T(s)Qx(s)ds + u^T(s)Ru(s)ds)\,.$$

(2) **Infinite horizon optimal control:** u^* minimizes the cost functional

$$J(u, x_0) = \limsup_{t_1\to\infty} J(u, x_0, t_1)\,.$$

Contribution of the paper We show that the optimal control problem above has a solution for a given x_0 if and only if (1) is stabilizable from x_0. We also establish a link between DAEs and classical LTI system, which allows us to re-use existing theory and understand better the notion of stabilizability for DAEs. More precisely, we solve the infinte-horizon LQ control problem by transforming it to a classical LTI optimal control problem. To this end we show that any solution of DAE (1) can be represented as an output of a linear time-invariant system (A_l, B_l, C_l, D_l): (x, u) satisfies (1), if there exists an input v such that: $\dot{p} = A_lp + B_lv$ and $(x^T, u^T)^T = C_lp + D_lv$. The corresponding LTI (A_l, B_l, C_l, D_l) can be obtained by using geometric theory of LTI systems [2, 22]: namely, by computing the largest output nulling controlled invariant subspace. In fact, we show the existence of a class of LTIs whose outputs are solutions of DAEs and we also show that all LTIs corresponding to the same DAE are feedback equivalent. The finite (infinite) horizon LQ control problem for (1) can then be reformulated as a finite (infinite) horizon classical LQ control problem. In particular, the corresponding infinite horizon LQ control problem has a solution, if $\mathscr{S} = (A_l, B_l, C_l, D_l)$ is stabilizable from p_0, where p_0 is the initial state of $\mathscr{S}$ which corresponds to x_0. In turn, the latter is equivalent to stabilizability of (1) from x_0.

Note that we do not restrict E, A, B in order to show the correspondence above: namely, the DAE (1) can be represented as an output of an associated LTI system for any E, A, B.

Motivation DAEs have a wide range of applications: namely, without claiming completeness, we mention robotics [17], cyber-security [15] and modelling [14]. The

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motivation for considering the optimal control problem above is twofold. First, it is interesting on its own right. Second, from [30] it follows that the solution of the optimal control problem yields a minimax observer for the DAE with uncertain but bounded input and incomplete and noisy observations. In turn, the latter can be used for example for state estimation of partial differential equations [27, 28].

Related work To the best of our knowledge, the main result of this paper is new. An extended version of this paper is available at [16] and its preliminary version appeared in [29]. With respect to [29] the main difference is that we included detailed proofs, and provided necessary and sufficient conditions for existence of a solution for the infinite horizon optimal control problem. The solution of the finite horizon optimal control problem was already presented in [26]. In contrast to [26] we consider the infinite horizon case too, and we present a more detailed proof for the finite horizon case by using the same framework as for the infinite horizon case. That is, we present a generic procedure for reducing the finite horizon optimal control problem to a classical finite horizon LQ problem for LTI systems. This procedure contains the reduction algorithm proposed in [26] as a particular case.

The literature on optimal control for DAE is vast. For an overview we refer the reader to [7, 12] and the references therein. To the best of our knowledge, the most relevant references are [3, 18, 19, 21, 23, 25]. In [3, 23] only regular DAEs were considered. In contrast, we allow any DAE which is stabilizable from the designated initial state. The infinite-horizon LQ control problem for non-regular DAE was also addressed in [25], however there it is assumed that the DAE has a solution from any initial state. In terms of results, [18–21] seem to be the closest to our paper. However, they neither imply nor they are implied by our results. The main differences are as follows.

(1) In [18–21] the cost function does not have the final cost term $x^T(t_1)Q_0x(t_1)$. Note that the presence of this terms is indispensable to make the duality work (see [30, Theorem 1] for the further details), when the results of this paper are applied to the minimax observer design problem [30].
(2) We consider existence of a solution from a particular initial condition, as opposed to [18, 19, 21].
(3) We present necessary and sufficient conditions for existence of an optimal solution. In contrast, [18–21] present just sufficient conditions. Moreover, those sufficient conditions appear to be more restrictive than the ones of this paper. This is not surprising, since [18–21] consider a different cost function and impose different additional restrictions on the optimal solution. As a result, we can weaken some of the conditions imposed in [18–21].
(4) We require only Ex to be absolutely continuos, not x. This leads to subtle but important technical differences in the solution.
(5) Finally, similarly to [18, 21], we solve the LQ problem by reducing it to an LTI LQ problem. Unlike in [18, 21], where Silverman's algorithm was used, we obtain corresponding LTI through geometric arguments. The construction of [18, 21] then becomes a particular case of our construction, if we restrict x to be only absolutely continuos.

Optimal control of non-linear and time-varying DAEs was also addressed in the literature, see [8–10]. However, they do not seem to be directly applicable to the problem of the current paper, as they look at finite horizon optimal control problems. Furthermore, the cited papers focus more on the (very challenging) problem of existence of a solution rather than on algorithms

The idea of presenting solutions of general DAEs as outputs of LTIs appeared in [18, 21, 25, 26]. In [18, 21, 26] an algorithm was presented to compute such an LTI. In this paper we show that a whole family of such LTIs can be obtained using classical results from geometric control theory, and thus the algorithms of [18, 21, 26] are specific instances of the classical algorithms of geometric control theory. The construction of [25] represents a special case of the results of this paper.

Note that relating DAEs with ordinary differential equations via Weierstrass canonical forms is a classical trick. However, this trick requires either smoothness constraints on admissible inputs or usage of distributions. Despite the superficial difference, there is a connection between our approach and that of based on quasi-Weistrass canonical forms [4]. We postpone the detailed discussion until Remark 7. There is a long history of geometric ideas in DAEs, see for example [1, 13] and the references in the surveys [5, 12]. However, to the best of our knowledge, these ideas were not used to relate LTIs and DAEs.

Outline of the paper This paper is organized as follows. Subsection 1.1 contains notations, section 2 describes the mathematical problem statement, section 3 presents main results of the paper.

1.1 Notation

$\mathcal{I}_n$ denotes the $n \times n$ identity matrix; for an $n \times n$ matrix S, $S > 0$ means $x^T S x > 0$ for all $x \in \mathbb{R}^n$, $\|S\|$ denotes the Frobenious norm; F^+ denotes the pseudoinverse of the matrix F. $I := [0, t]$ for $t \le +\infty$ and $L^2(I, \mathbb{R}^n)$ denotes the space of all square-integrable functions $f : I \to \mathbb{R}^n$, $L^2_{loc}(I, \mathbb{R}^n) = \{f \in L^2(I^1, \mathbb{R}^n), \forall I^1 \subsetneq I, I^1 \text{ is compact }\}$. To simplify the notation we will often write $L^2(0, t)$ and $L^2_{loc}(0, t)$ referring to $L^2(I, \mathbb{R}^n)$ and $L^2_{loc}(I, \mathbb{R}^n)$ respectively. Note that $L^2(0, t) = L^2_{loc}(0, t)$ if $t < +\infty$. Finally, $f|_A$ stands for the restriction of a function f onto a set A.

2 Problem statement

Consider the DAE Σ:

$$\frac{dEx}{dt} = Ax(t) + Bu(t) \text{ and } Ex(0) = Ex_0 \,. \qquad (2)$$

Here $A, E \in \mathbb{R}^{c\times n}$, $B \in \mathbb{R}^{c\times m}$ and $x_0 \in \mathbb{R}^n$ is a fixed initial state.

Notation 1 ($\mathscr{D}_{x_0}(t_1)$ and $\mathscr{D}_{x_0}(\infty)$) . *For any $t_1 \in [0, +\infty]$ denote by I the interval $[0, t_1] \cap [0, +\infty)$ and denote by $\mathscr{D}_{x_0}(t_1)$ the set of all pairs $(x, u) \in L^2_{loc}(I, \mathbb{R}^n) \times L^2_{loc}(I, \mathbb{R}^m)$ such that Ex is absolutely continuous and (x, u) satisfy (2) almost everywhere. A state x_0 will be called* differentiably consistent *if $\mathscr{D}_{x_0}(t_1) \neq \emptyset$. The set of differentiably consistent initial states is denoted by $\mathcal{V}_c$.*

We stress that there may exist an initial state x_0 such that $\mathcal{D}_{x_0}(t_1)$ is empty for some $t_1 \in [0, +\infty]$. Note that our definition of differentiable consistency is slightly different from the one used in the literature, see [5]: namely, for an initial state x_0 to be consistent, we only require that there exists a locally integrable solution x such that $Ex(0) = Ex_0$; we do not require that $x(0) = x_0$ or impose continuity (differentiability) of x.

Next, we formulate the optimal control problem already mentioned in the introduction.

Take symmetric $R \in \mathbb{R}^{m\times m}, Q, \in \mathbb{R}^{n\times n}$, $Q_0 \in \mathbb{R}^{c\times c}$ and assume that $R > 0, Q > 0$ $Q_0 \geq 0$. For any initial state $x_0 \in \mathbb{R}^n$, and any trajectory $(x, u) \in \mathscr{D}_{x_0}(t)$, $t > t_1$ define the cost functional

$$\begin{aligned} J(x, u, t_1) &= x(t_1)^T E^T Q_0 Ex(t_1) + \\ &+ \int_0^{t_1} (x^T(s)Qx(s) + u^T(s)Ru(s))ds. \end{aligned} \qquad (3)$$

Problem 1 (Finite-horizon optimal control) *The problem of finding $(x^*, u^*) \in \mathscr{D}_{x_0}([0, t_1])$ such that:*

$$J(x^*, u^*, t_1) = \inf_{(x,u)\in\mathscr{D}_{x_0}([0,t_1])} J(x, u, t_1) < +\infty$$

is called the finite-horizon optimal control problem *and (x^*, u^*) is called the solution of the finite-horizon (optimal) control problem.*

Problem 2 (Infinite horizon optimal control) *For every $(x, u) \in \mathscr{D}(\infty)$, define*

$$J(x, u) = \limsup_{t_1\to\infty} J(x, u, t_1) \,.$$

The infinite horizon (optimal) control problem for (2) and initial state x_0 is the problem of finding (x^*, u^*) such that $(x^*, u^*) \in \mathscr{D}_{x_0}(\infty)$ and

$$J(x^*, u^*) = \limsup_{t_1\to\infty} \inf_{(x,u)\in\mathscr{D}_{x_0}(t_1)} J(x, u, t_1) < +\infty \,. \quad (4)$$

The pair (x^*, u^*) will be called the solution of the infinite horizon (optimal) control problem for the initial state x_0.

Remark 1 *The proposed formulation of the infinite horizon control problem is not the most natural one. We could have also required the $(x^*, u^*) \in \mathcal{D}(\infty)$ to satisfy $J(x^*, u^*) = \inf_{(x,u)\in\mathscr{D}(\infty)} J(x, u)$. The latter means that the cost induced by (x^*, u^*) is the smallest among all the trajectories (x, u) which are defined on the whole time axis. It is easy to see that formulation above implies that $J(x^*, u^*) = \inf_{(x,u)\in\mathscr{D}(\infty)} J(x, u)$. Another option is to use* $\lim$ *instead of* $\limsup$ *in the definition of $J(x^*, u^*)$ and in (4). In fact, the solution we are going to present remains valid if we replace* $\limsup$ *by* $\lim$.

Remark 2 (Solution as a feedback) *In the sequel we will show that the solution of the optimal control problem (x^*, u^*) is such that $u^*(t) = K(t)x^*(t)$ for some state feedback matrix $K(t)$ of suitable dimensions. Note, however, that the feedback law does not determine the control input uniquely, since DAE (2) may admit several solutions starting from the same initial state. If the DAE has at most one solution from any initial state, in particular, if the DAE is regular, then the feedback law above determines the optimal trajectory x^* uniquely.*
In the general case, one could follow the behavioral interpretation adopted by [24]. There, the interconnection of a feedback controller $u = Kx$, where K is a constant matrix, with the original system was interpreted as the following DAE

$$\frac{d}{dt}(\hat{E}\hat{x}) = \hat{A}\hat{x}$$

where $\hat{E} = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}$, $\hat{A} = \begin{bmatrix} A & 0 \\ K & I \end{bmatrix}$, *and* $\hat{x}(t) = (x(t), u(t))$.

In order to state the conditions under which Problem 2 has a solution, we will need the notion of stabilizability.

Definition 1 (Stabilizability) *The DAE (2) is said to be stabilizable for a differentiably consistent initial state $x_0 \in \mathbb{R}^n$, if there exists $(x, u) \in \mathcal{D}_{x_0}(\infty)$ such that $\lim_{t\to\infty} Ex(t) = 0$. The DAE (2) is said to be stabilizable, if it is stabilizable for all $x_0 \in \mathbb{R}^n$ such that x_0 is differentiably consistent.*

In the sequel we will show that the solution (x^*, u^*) of the infinite horizon optimal control problem is such that $u^* = Kx^*$ and $\lim_{t\to\infty} Ex^*(t) = 0$ and that stabilizability from x_0 is necessary and sufficient for existence of a solution to the optimal control problem. Hence, if the DAE is stabilizable from x_0, then there is a feedback control law such that the closed-loop system has a stable trajectory.

3 Main results

In §3.1 we present the construction of an LTI system whose outputs are solutions of the original DAE. In §3.2 we reformulate the optimal control problem as a linear quadratic infinite horizon control problem for LTIs. The solution of the latter problem yields a solution to the original control problem.

3.1 DAE systems as solutions to the output zeroing problem

We provide a complete characterization of all solutions of (2) as outputs of an LTI system.

Theorem 1 *Consider the DAE system* (2). *There exists a linear system* $\mathscr{S} = (A_l, B_l, C_l, D_l)$ *defined on state-space* $X = \mathbb{R}^{\hat{n}}$, *input space* $U = \mathbb{R}^k$ *and output space* $\mathbb{R}^{n+m}$, *and a linear subspace* $\mathcal{X} \subseteq \mathbb{R}^c$ *such that the following holds.*

(1) $\mathrm{Rank} D_l = k$.
(2) Let C_s *and* D_s *be the matrices formed by the first* n *rows of* C_l *and* D_l *respectively. Then* $ED_s = 0$, $\mathrm{Rank} EC_s = \hat{n}$, $\mathcal{X} = \mathrm{im} EC_s$.
(3) The state x_0 *is differentiably consistent if and only if* $Ex_0 \in \mathcal{X}$.
(4) Fix $t_1 \in [0, +\infty]$ *and set* $I = [0, t_1] \cap [0, +\infty)$. *For any* $g \in L^2_{loc}(I, \mathbb{R}^k)$, *let*

$$\begin{aligned} v &= v_{\mathscr{S}}(v_0, g) : I \to \mathbb{R}^{\hat{n}}, \\ \nu &= \nu_{\mathscr{S}}(v_0, g) : I \to \mathbb{R}^{n+m} \end{aligned} \tag{5}$$

be the state and output trajectory of $\mathscr{S}$ *which corresponds to the initial state* v_0 *and input* g, *i.e.*

$$\begin{aligned} \dot{v} &= A_l v + B_l g \text{ and } v(0) = v_0 \\ \nu &= C_l v + D_l g \,. \end{aligned} \tag{6}$$

Define the map $\mathcal{M} = (EC_s)^+ : \mathbb{R}^c \to \mathbb{R}^{\hat{n}}$. *Assume that* $Ex_0 \in \mathcal{X}$. *Then the following holds.*
(a) If $(x, u) \in \mathcal{D}_{x_0}(t_1)$, *then there exists an input* $g \in L^2_{loc}(I, \mathbb{R}^k)$, *such that*

$$\nu_{\mathscr{S}}(\mathcal{M}(Ex_0), g)(t) = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \text{ for } t \in I \text{ a.e.} \tag{7}$$

(b) Conversely, for any $g \in L^2_{loc}(I, \mathbb{R}^k)$, *there exists* $(x, u) \in \mathcal{D}_{x_0}(t_1)$ *such that* $\nu_{\mathscr{S}}(\mathcal{M}(Ex_0), g)(t) = (x^T(t), u^T(t))^T$ *for all* $t \in I$.
(c) If (7) *holds for some* $(x, u) \in \mathcal{D}_{x_0}(t_1)$, *then* $\mathcal{M}(Ex(t)) = v_{\mathscr{S}}(\mathcal{M}(Ex(0), g)(t) =: v(t)$ *for all* $t \in I$, *and* $g(t) = D_l^+(\begin{bmatrix} x(t) \\ u(t) \end{bmatrix} - C_l v(t))$ *for* $t \in I$ *a.e.*

PROOF. [Proof of Theorem 1] There exist suitable nonsingular matrices S and T such that

$$SET = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \tag{8}$$

where $r = \mathrm{Rank} E$. Let

$$SAT = \begin{bmatrix} \widetilde{A} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \; SB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

be the decomposition of E, A, B such that $\widetilde{A} \in \mathbb{R}^{r \times r}$ and $B_1 \in \mathbb{R}^{r \times m}$. Define

$$G = \begin{bmatrix} A_{12}, \; B_1 \end{bmatrix}, \quad \widetilde{D} = \begin{bmatrix} A_{22}, \; B_2 \end{bmatrix} \text{ and } \widetilde{C} = A_{21}.$$

Consider the following linear system

$$\mathcal{S} \begin{cases} \dot{p} = \widetilde{A} p + G q \\ z = \widetilde{C} p + \widetilde{D} q \end{cases}. \tag{9}$$

Now, it is easy to see that if (x, u) solves (2) then (p, q) defined by $T^{-1}x = (p^T, q_1^T)^T$, $q = (q_1^T, u^T)^T$, $q_1 \in \mathbb{R}^{n-r}$ solve the linear system (9) and the output z is zero almost everywhere.

Recall from [22, Definition 7.8] the concept of a weakly unobservable subspace of a linear system. I.e., an initial state $p(0) \in \mathbb{R}^r$ of $\mathcal{S}$ is *weakly unobservable*, if there exists an input function $q \in L^2_{loc}([0, +\infty), \mathbb{R}^k)$ such that the resulting output function z of $\mathcal{S}(\Sigma)$ equals zero, i.e. $z(t) = 0$ for almost all $t \in [0, +\infty)$. Following [22], let us denote the set of all weakly unobservable states by $\mathcal{V}(\mathcal{S})$. Recall from [22, Section 7.3], $\mathcal{V}(\mathcal{S})$ is a vector space and in fact it can be computed. Moreover, if $p(0) \in \mathcal{V}(\mathcal{S})$ and for the particular choice of q, $z = 0$, then $p(t) \in \mathcal{V}(\mathcal{S})$ for all $t \geq 0$.

Let $I = [0, t]$ or $I = [0, +\infty)$. Let $q \in L^2_{loc}(I, \mathbb{R}^{n-r+m})$ and let $p_0 \in \mathbb{R}^r$. Denote by $p(p_0, q)$ and $z(p_0, q)$ the state and output trajectory of (9) which corresponds to the initial state p_0 and input q. For technical purposes we will need the following easy extension of [22, Theorems 7.10–7.11].

Theorem 2 *(1)* $\mathcal{V} = \mathcal{V}(\mathcal{S})$ *is the largest subspace of* $\mathbb{R}^r$ *for which there exists a linear map* $\widetilde{F} : \mathbb{R}^r \to \mathbb{R}^{m+n-r}$ *such that*

$$(\widetilde{A} + G\widetilde{F})\mathcal{V} \subseteq \mathcal{V} \text{ and } (\widetilde{C} + \widetilde{D}\widetilde{F})\mathcal{V} = 0 \tag{10}$$

(2) Let $\widetilde{F}$ *be a map such that* (10) *holds for* $\mathcal{V} = \mathcal{V}(\mathcal{S})$. *Let* $L \in \mathbb{R}^{(m+n-r) \times k}$ *for some* k *be a matrix such that* $\mathrm{im} L = \ker \widetilde{D} \cap G^{-1}(\mathcal{V}(\mathcal{S}))$ *and* $\mathrm{Rank} L = k$.

For any interval $I=[0,t]$ or $I=[0,+\infty)$, and for any $p_0 \in \mathbb{R}^r$, $q \in L^2_{loc}(I,\mathbb{R}^k)$,

$$z(p_0,q)(t)=0 \text{ for } t\in I \text{ a.e.}$$

if and only if $p_0 \in \mathcal{V}$ and there exists $g \in L^2_{loc}(I,\mathbb{R}^k)$ such that:

$$q(t)=\widetilde{F}p(p_0,q)(t)+Lg(t) \text{ for } t\in I \text{ a.e.}$$

PROOF. [Proof of Theorem 2] Part 1 is a reformulation of [22, Theorem 7.10]. For $I=[0,+\infty)$, Part 2 is a restatement of [22, Theorem 7.11]. For $I=[0,t_1]$, the proof is similar to [22, Theorem 7.11].

We are ready now to finalize the proof of Theorem 1. The desired linear system $\mathscr{S}=(A_l,B_l,C_l,D_l)$ may be obtained as follows. Consider the linear system below.

$$\begin{aligned}
&\dot p = (\widetilde{A}+G\widetilde{F})p+GLg\\
&(x^T,u^T)^T=\bar{C}p+\bar{D}g\\
&\bar{C}=\begin{bmatrix}T & 0\\ 0 & I_m\end{bmatrix}\begin{bmatrix}I_r\\ \widetilde{F}\end{bmatrix} \text{ and } \bar{D}=\begin{bmatrix}T & 0\\ 0 & I_m\end{bmatrix}\begin{bmatrix}0\\ L\end{bmatrix}.
\end{aligned}$$

Consider a basis $b_1,\ldots,b_r$ of $\mathbb{R}^r$ such that $b_1,\ldots,b_{\hat n}$ span $\mathcal{V}$. Let $\mathscr{R}=\begin{bmatrix}b_1, & \ldots, & b_r\end{bmatrix}^{-1}$ be the corresponding basis transformation. It then follows that $\mathscr{R}(\mathcal{V})=\operatorname{im}\begin{bmatrix}I_{\hat n} & 0\\ 0 & 0\end{bmatrix}$. Set $A_l=\begin{bmatrix}I_{\hat n} & 0\end{bmatrix}\mathscr{R}(\widetilde{A}+G\widetilde{F})\mathscr{R}^{-1}\begin{bmatrix}I_{\hat n}\\ 0\end{bmatrix}$, $C_l=\bar{C}\mathscr{R}^{-1}\begin{bmatrix}I_{\hat n}\\ 0\end{bmatrix}$, $B_l=\begin{bmatrix}I_{\hat n} & 0\end{bmatrix}\mathscr{R}GL$, and $D_l=\bar{D}$. That is, A_l,B_l,C_l are the matrix representations in the basis $b_1,\ldots,b_{\hat n}$ of the linear maps $A_l=(\widetilde{A}+G\widetilde{F})|_{\mathcal{V}}:\mathcal{V}\to\mathcal{V}$, $C_l=\bar{C}|_{\mathcal{V}}:\mathcal{V}\to\mathbb{R}^{n\times m}$, $B_l=GL:\mathbb{R}^k\to\mathcal{V}$. Define

$$\mathcal{X}=\{ET\begin{bmatrix}p\\0\end{bmatrix}\mid p\in\mathcal{V}\}.$$

It is easy to see that this choice of (A_l,B_l,C_l,D_l) and $\mathcal{X}$ satisfies the conditions of the theorem.

The proof of Theorem 1 is constructive and yields an algorithm for computing (A_l,B_l,C_l,D_l) from (E,A,B). This prompts us to introduce the following terminology.

Definition 2 *A linear system $\mathscr{S}=(A_l,B_l,C_l,D_l)$ described in the proof of Theorem 1 is called the linear system associated with the DAE* (2)*, and the map $\mathcal{M}$ is called the corresponding state map.*

Remark 3 *Notice that the dimension $\hat n$ of the associated linear system satisfies $\hat n \le \operatorname{Rank}E \le \max\{c,n\}$.*

Remark 4 *It is easy to see from the construction that $\begin{bmatrix}C_l & D_l\end{bmatrix}$ is full column rank, i.e. it represents an injective linear map. Indeed, if $C_lp+D_lg=0$, then $C_sp+D_sg=0$ and hence, by using $ED_s=0$, $E(C_sp+D_sg)=EC_sp=0$. Since EC_s is full column rank, it follows that $p=0$. Hence, $C_lp+D_lg=D_lg=0$, but D_l is full column rank and hence $g=0$. That is, $C_lp+D_lg=0$ implies $p=0,g=0$.*

Remark 5 (Comparison with [26]) *The system (A^s_1,A^s_2) described in [26, Proposition 3] is related to a linear associated system as follows. If we identify $\widetilde{A}=A^0_1$, $G=A^0_2$, $\widetilde{C}=A^0_3$, $\widetilde{D}=A^0_4$, and define $\tilde{F}$ as in* (10) *then $\mathcal{V}(S)=\operatorname{im}P^s_0$, and $A^s_1|_{\mathcal{V}(S)}=(\widetilde{A}+G\widetilde{F})|_{\mathcal{V}(S)}$, $A^s_2L=GL$. Moreover, if (p,w) is a solution to $\dot p=(\widetilde{A}+G\widetilde{F})p+GLw$ such that $p(0)\in\mathcal{V}(S)$, then $\hat p=p$ and $\hat q=Lw$ is a solution to [26, eq. (22)]. For the proof see [16]*

Remark 6 (Impulse controllability) *Recall from [5] that the DAE* (2) *is impulse controllable, if for any matrix Z such that $\operatorname{im}Z=\ker E$, $d=\operatorname{Rank}\begin{bmatrix}E, & AZ, & B\end{bmatrix}=\operatorname{Rank}\begin{bmatrix}E, & A, & B\end{bmatrix}$. Let S,T be the non-singular matrices from the proof of Theorem 1. Choose $Z=T\begin{bmatrix}0 & 0\\ 0 & I_{n-r}\end{bmatrix}$, By multiplying $\begin{bmatrix}E, & AZ, & B\end{bmatrix}$ by S from the left and by $X=\operatorname{diag}(T,I_n,I_m)$ from right, and using the notation of the proof of Theorem 1,*

$$d=\operatorname{Rank}S\begin{bmatrix}E, & AZ, & B\end{bmatrix}X=r+\operatorname{Rank}\widetilde{D}$$

By multiplying $\begin{bmatrix}E, & A, & B\end{bmatrix}$ by S from the left and by $Y=mathrmdiag(T,T,I_m)$ from the right,

$$d=\operatorname{Rank}S\begin{bmatrix}E, & A, & B\end{bmatrix}Y=r+\operatorname{Rank}\begin{bmatrix}\widetilde{C}, & \widetilde{D}\end{bmatrix}.$$

Hence, $\operatorname{Rank}\begin{bmatrix}\widetilde{C}, & \widetilde{D}\end{bmatrix}=\operatorname{Rank}\widetilde{D}$, *i.e.* $\operatorname{im}\widetilde{C}\subseteq\operatorname{im}\widetilde{D}$. *In this case , $\mathcal{V}(\mathcal{S})=\mathbb{R}^r$, L is any matrix such that* $\operatorname{im}L=\ker\widetilde{D}$ *and $\widetilde{F}=-\widetilde{D}^{+}\widetilde{C}$.*

Note that the linear system associated with (E,A,B) is not unique, since the choice of the matrices $S,T,\widetilde{F},L,\mathscr{R}$ in the proof of Theorem 1 is not unique. However, we can show that all associated linear systems are feedback equivalent.

Definition 3 (Feedback equivalence) *Two linear systems $\mathscr{S}_i=(A_i,B_i,C_i,D_i)$, $i=1,2$ are said to be* feedback equivalent, *if there exist a linear state feedback matrix K and a non-singular square matrices U, T such that $(T(A_1+B_1K)T^{-1}, TB_1U, (C_1+D_1K)T^{-1}, D_1U) = \mathscr{S}_2$. We will call (T,K,U) feedback equivalence from $\mathscr{S}_1$ to $\mathscr{S}_2$.*

Lemma 1 *Let $\mathscr{S}_i$, $i=1,2$ be two linear systems which are obtained from the proof of Theorem 1 and let $\mathcal{M}_i$, $i=1,2$ be the corresponding state maps. Then $\mathscr{S}_1$ and $\mathscr{S}_2$ are feedback equivalent. Moreover, if (T,K,U) is the corresponding feedback equivalence, then $T\mathcal{M}_1 = \mathcal{M}_2$.*

The proof of Lemma 1 can be found in the appendix.

Lemma 2 *If there exists $t_1 > 0$ such that $\mathcal{D}_{x_0}(t_1) \neq \emptyset$, then $\mathcal{D}_{x_0}(\infty) \neq \emptyset$. In other words, if x_0 is differentiably consistent, then there exists a solution $(x,u) \in \mathcal{D}_{x_0}(\infty)$.*

PROOF. Let $\mathscr{S} = (A_l, B_l, C_l, D_l)$ be the linear system associated with the DAE. From Theorem 1 it then follows that if x_0 is differentiably consistent for some $t_1 > 0$, then any solution $(x,u) \in \mathcal{D}_{x_0}(t_1)$ arises as the output of $\mathscr{S}$ corresponding to some input g and the initial state $\mathcal{M}(Ex_0)$. This solution is defined for all $t > 0$ and so belongs to $(x,u) \in \mathcal{D}_{x_0}(\infty)$. In particular, if $g = 0$, then the output (x,u) of $\mathscr{S}$ from the initial state $\mathcal{M}(Ex_0)$ is smooth.

DAE. Recall from [22] the notion of the stabilizability subspace $\mathcal{V}_g$ of a linear system $\mathscr{S} = (A,B,C,D)$: $\mathcal{V}_g$ is the set of all initial states x_0 of $\mathscr{S}$, for which there exists an input u such that the corresponding state trajectory x starting from x_0 has the property that $\lim_{t\to\infty} x(t) = 0$.

Lemma 3 *The DAE (2) is stabilizable from x_0 if and only if $\mathcal{M}(Ex_0)$ belongs to the stabilizability subspace $\mathcal{V}_g$ of $\mathscr{S}$ for every associated linear system $\mathscr{S} = (A_l, B_l, C_l, D_l)$ and the corresponding map $\mathcal{M}$. In particular, the DAE is stabilizable if and only if $\mathscr{S}$ is stabilizable.*

PROOF. Since feedback equivalence preserves stabilizability subspaces and all associated linear systems are feedback equivalent, existence of an associated linear systems which is stabilizable from $\mathcal{M}(Ex_0)$ is equivalent to saying that all associated linear systems are stabilizable from $\mathcal{M}(Ex_0)$. Assume that one of the associated linear systems $\mathscr{S} = (A_l, B_l, C_l, D_l)$ is stabilizable from $\mathcal{M}(Ex_0)$. Let $\mathcal{V}_g$ be the stabilizability subspace of $\mathscr{S}$. It then follows from [22] that $\mathcal{M}(Ex_0) \in \mathcal{V}_g$, and there exists a feedback F_l such that the restriction of $A_l + B_lF_l$ to $\mathcal{V}_g$ is stable and hence for any $v_0 = \mathcal{M}(Ex_0)$, there exists a state $\dot{v} = (A_l + B_lF_l)v$ with $v(0) = v_0$, such that $\lim_{t\to\infty} v(t) = 0$. Consider the output $(x^T, u^T)^T = (C_l + D_lF_l)v$. It then follows that $Ex(0) = Ex_0$ and (x,u) is a solution of the DAE. Moreover, $\lim_{t\to\infty} x(t) = \lim_{t\to\infty}(C_s + D_sF_l)v(t) = 0$. That is, for this particular u, $\lim_{t\to\infty} Ex(t) = 0$ and thus the DAE is stabilizable from x_0. Conversely, assume that the DAE is stabilizable from x_0, and choose any associated linear system (A_l, B_l, C_l, D_l). Let $(x,u) \in \mathcal{D}_{x_0}(\infty)$ be a solution of the DAE such that $\lim_{t\to\infty} Ex(t) = 0$. It then follows that there exist an input g such that $\dot{v} = A_lv + B_lg$, $(x^T, u^T)^T = C_lv + D_lg$, $v(0) = v_0 = \mathcal{M}(Ex_0)$ and $\mathcal{M}(Ex(t)) = v(t)$. In particular, $\lim_{t\to\infty} v(t) = \mathcal{M}\lim_{t\to\infty} Ex(t) = 0$. That is, there exists an input g, such that the corresponding state trajectory v of $\mathscr{S}$ starting from $\mathcal{M}(Ex_0)$ converges to zero. But this is precisely the definition of stabilizability of (A_l, B_l, C_l, D_l) from $\mathcal{M}(Ex_0)$.

Remark 7 *Recall from [5] that the augmented Wong sequence is defined as follows $\mathcal{V}_0 = \mathbb{R}^n$, $\mathcal{V}_{i+1} = A^{-1}(E\mathcal{V}_i + \mathrm{im}B)$, and that the limit $\mathcal{V}^* = \bigcap_{i=0}^{\infty} \mathcal{V}_i$ is achieved in finite number of steps: $\mathcal{V}^* = \mathcal{V}_k$ for some $k \in \mathbb{N}$. It is not difficult to see that $\mathcal{V}(\mathcal{S})$ from the proof of Theorem 1 correspond to the limit $\mathcal{V}_*$ of the augmented Wong sequence $\mathcal{V}_i$ for the DAE (2): $\mathcal{V}^* = \{(p, Fp + Lq)^T \mid p \in \mathcal{V}(\mathcal{S})\}$. Hence, if (A_l, B_l, C_l, D_l) is as in Theorem 1, then $\mathcal{V}^* = \mathrm{im}\begin{bmatrix} C_s & D_s \end{bmatrix}$. In [4] a relationship between the quasi-Weierstrass form of regular DAEs and and space $\mathcal{V}^*$ for $B = 0$ was established. This indicates that there might be a deeper connection between quasi-Weierstrass forms and associated linear systems. The precise relationship remains a topic of future research.*

3.2 Solution of the optimal control problem for DAE

We apply Theorem 1 in order to solve the Problem 2. Let $\mathscr{S} = (A_l, B_l, C_l, D_l)$ be a linear system associated with Σ and let $\mathcal{M}$ and C_s be defined as in Theorem 1. Consider the following linear quadratic control problem. For every initial state v_0, for every interval I containing $[0, t_1]$ and for every $g \in L^2_{loc}(I, U)$ define a cost functional $J(v_0, g, t)$:

$$\mathscr{J}(v_0, g, t_1) = v^T(t_1)E^TC_s^TQ_0EC_sv(t_1) + \\ + \int_0^{t_1} \nu^T(t) \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \nu(t)dt \\ \dot{v} = A_lv + B_lg \text{ and } v(0) = v_0 \\ \nu = C_lv + D_lg.$$

For any $g \in L^2_{loc}([0, +\infty))$ and $v_0 \in \mathbb{R}^{\hat{n}}$, define

$$\mathscr{J}(v_0, g) = \limsup_{t_1\to\infty} \mathscr{J}(v_0, g, t_1).$$

In our next theorem we prove that the problem of minimizing the cost function J for (2) is equivalent to minimizing the cost function $\mathscr{J}$ for the associated LTI $\mathscr{S}$.

Theorem 3 *Consider the associated LTI $\mathscr{S}$ and recall that for $g \in L^2_{loc}(I)$, $t_1 \in [0, +\infty]$, $I = [0, t_1] \cap [0, +\infty)$, $\nu_{\mathscr{S}}(v_0, g)$ denotes the output trajectory of $\mathscr{S}$ which corresponds to the initial state v_0 and input g.*

(i) *Assume that x_0 is a differentiably consistent initial state of (1). For any $g \in L^2_{loc}(I)$ and for any $(x, u) \in \mathscr{D}_{x_0}(t_1)$ such that $(x^T, u^T)^T = \nu_{\mathscr{S}}(\mathcal{M}(Ex_0), g)$ a.e. ,*

$$\begin{aligned} J(x,u,t_1) &= \mathscr{J}(\mathcal{M}(Ex_0), g, t_1) \\ J(x,u) &= \mathscr{J}(\mathcal{M}(Ex_0), g) \end{aligned} \tag{11}$$

(ii) *For all $t_1 > 0$, $(x^*, u^*) \in \mathscr{D}_{x_0}(t_1)$ is a solution of the finite horizon optimal control problem if and only if there exists $g^* \in L^2(0, t_1)$ such that $({x^*}^T, {u^*}^T)^T = \nu_{\mathscr{S}}(\mathcal{M}(Ex_0), g^*)$ a.e. and*

$$\mathscr{J}(v_0, g^*, t_1) = \inf_{g \in L_2([0,t_1])} \mathscr{J}(v_0, g, t_1) < +\infty\,. \tag{12}$$

(iii) *The tuple $(x^*, u^*) \in \mathscr{D}_{x_0}(\infty)$ is a solution of the infinite horizon optimal control problem if and only if there exists an input $g^* \in L^2_{loc}(0, +\infty)$ such that $({x^*}^T, {u^*}^T)^T = \nu_{\mathscr{S}}(\mathcal{M}(Ex_0), g^*)$ a.e., and*

$$\mathscr{J}(v_0, g^*) = \limsup_{t_1 \to \infty} \inf_{g \in L^2(0,t_1)} \mathscr{J}(v_0, g, t_1) < +\infty. \tag{13}$$

PROOF. [Proof of Theorem 3] Equation (11) follows by routine manipulations and by noticing that the first n rows of $C_l v(t) + D_l g(t)$ equal $C_s v(t) + D_s g(t)$ and as $ED_s = 0$ (see Theorem 1), $E(C_s v(t) + D_s g(t)) = EC_s v(t)$. The rest of theorem follows by noticing that any element of $\mathcal{D}_{x_0}(t_1)$ or $\mathcal{D}_{x_0}(\infty)$ arises as an output of $\mathscr{S}$ for some $g \in L^2_{loc}(I, \mathbb{R}^k)$.

Next theorem is based on classical results (see [11]) and provides the solution of the finite horizon optimal control problem (Problem 1.).

Theorem 4 *Assume that x_0 is a differentially consistent initial state of (1). The finite horizon optimal control problem has a unique solution of the form*

$$\begin{aligned} ({x^*}^T(t), {u^*}^T(t))^T &= (C_l - D_l K(t_1 - t))v(t) \\ \dot{v}(t) &= (A_l - B_l K(t_1 - t))v(t) \text{ and } \ v(0) = \mathcal{M}(Ex_0)\,. \end{aligned} \tag{14}$$

where $P(t)$ and $K(t)$ satisfy the following Riccati differential equation

$$\begin{aligned} \dot{P}(t) &= A_l^T P(t) + P(t) A_l - K^T(t)(D_l^T S D_l) K(t) + C_l^T S C_l \\ P(0) &= (EC_s)^T Q_0 EC_s, \quad S = \operatorname{diag}(Q, R) \\ K(t) &= (D_l^T S D_l)^{-1}(B_l^T P(t) + D_l^T S C_l) \end{aligned} \tag{15}$$

where C_s is defined as in Theorem 1. Moreover, the optimal value of the cost function is

$$J(x^*, u^*, t_1) = (\mathcal{M}(Ex_0))^T P(t_1)(\mathcal{M}(Ex_0)). \tag{16}$$

PROOF. Let us first apply the feedback transformation $g = \hat{F}v + Uw$ to $\mathscr{S} = (A_l, B_l, C_l, D_l)$ with $U = (D_l^T S D_l)^{-1/2}$ and $\hat{F} = -(D_l^T S D_l)^{-1} D_l^T S C_l$, as described in [22, Section 10.5, eq. (10.32)]. Note that D_l is injective and hence U is well defined. Consider the linear system

$$\dot{v} = (A_l + B_l \hat{F})v + B_l U w \text{ and } v(0) = v_0 \tag{17}$$

For any $w \in L^2_{loc}(I)$, $I = [0, t_1]$ the state trajectory v of (17) equals the state trajectory of $\mathscr{S}$ for the input $g = \hat{F}v + Uw$ and initial state v_0. Moreover, from Theorem 2 it follows that all inputs g of $\mathscr{S}$ can be represented in such a way. Define now

$$\begin{aligned} &\widehat{\mathscr{I}}(v_0, w, t) = v^T(t) C_s^T E^T Q_0 E C_s v(t) + \\ &+ \int_0^t (v^T(t)(C_l + D_l\hat{F})^T S (C_l + D_l \hat{F}) v(t) + w^T(t) w(t)) dt, \end{aligned}$$

where v is a solution of (17), and C_s is defined as in the theorem's statement (so that C_s has n rows). It is easy to see that $\mathscr{J}(v_0, g, t) = \widehat{\mathscr{I}}(v_0, w, t)$ for $g = \hat{F}v + Uw$ and any initial state v_0 of $\mathscr{S}$.

Consider now the problem of minimizing $\widehat{\mathscr{I}}(v_0, w, t)$. The solution of this problem can be found using [11, Theorem 3.7]. Notice that (15) is equivalent to the Riccati differential equation described in [11, Theorem 3.7] for the problem of minimizing $\widehat{\mathscr{I}}(\hat{v}_0, w, t)$. Hence, by [11, Theorem 3.7], (15) has a unique positive solution P, and for the optimal input w^*, $g^* = \hat{F}v^* + Uw^* = -K(t_1 - t)v(t)$ satisfies (12) and $\dot{v}(t) = (A_l - B_l K(t_1 - t))v(t)$ and $v(0) = v_0$. From Theorem 3 and Part 4b of Theorem 1 it then follows that $({x^*}^T, {u^*}^T)^T = C_l v^* + D_l g^*$ is the solution of the Problem 1 and that (16) holds.

In our last theorem we present a solution to the infinite horizon optimal control problem (Problem 2). Let $\mathscr{S} = (A_l, B_l, C_l, D_l)$ be the LTI associated with (1). Let $\mathcal{V}_g$ denote the stabilizability subspace of $\mathscr{S}$. From [6] it then follows that $\mathcal{V}_g$ is A_l-invariant and $\operatorname{im} B_l \subseteq \mathcal{V}_g$. Hence, there exists a basis transformation T such that $T(\mathcal{V}_g) = \operatorname{im} \begin{bmatrix} I_l & 0 \\ 0 & 0 \end{bmatrix}$, $l = \dim \mathcal{V}_g$ and in this new basis,

$$TA_lT^{-1} = \begin{bmatrix} \hat{A}_g & \star \\ 0 & \star \end{bmatrix}, \quad TB_l = \begin{bmatrix} \hat{B}_g \\ 0 \end{bmatrix}, \quad C_l T^{-1} = \begin{bmatrix} \hat{C}_g^T \\ \star \end{bmatrix}^T,$$

$A_g \in \mathbb{R}^{l\times l}, \hat{B}_g \in \mathbb{R}^{l\times k}, \hat{C}_g \in \mathbb{R}^{n+m\times l}$. Denote by $\mathscr{S}_g = (\hat{A}_g, \hat{B}_g, \hat{C}_g, \hat{D}_g)$, where $D_g = D_l$.

Definition 4 (Stabilizable associated LTI) *We call $\mathscr{S}_g$ the stabilizable LTI associated with* (1) *and we call $\mathcal{M}_g = \begin{bmatrix} I_l & 0 \end{bmatrix} T\mathcal{M}$ the associated map.*

The LTI $\mathscr{S}_g$ represents the restriction of $\mathscr{S}$ to the subspace $\mathcal{V}_g$. It follows that $\mathscr{S}_g$ is stabilizable. Using Theorem 3 and the classical results on infinite horizon LQ control, we obtain the following.

Theorem 5 *The following are equivalent:*

- ***(i)*** *The infinite horizon optimal control problem is solvable for x_0*
- ***(ii)*** *$\mathcal{M}(Ex_0)$ belongs to the stabilizability subspace $\mathcal{V}_g$ of the associated LTI $\mathscr{S}$*
- ***(iii)*** $\limsup_{t_1\to\infty} \inf_{(x,u)\in\mathcal{D}_{x_0}(t_1)} J(x,u,t_1) < +\infty$

Consider the stabilizable LTI $\mathscr{S}_g = (A_g, B_g, C_g, D_g)$ and the map $\mathcal{M}_g$. If $\mathcal{M}(F^T\ell) \in \mathcal{V}_g$, then there exists a unique positive definite matrix P such that:

$$\begin{aligned} 0 &= PA_g + A_g^T P - K^T(D_g^T S D_g)K + C_g^T S C_g\,, \\ K &= (D_g^T S D_g)^{-1}(B_g^T P + D_g^T S C_g) \text{ and } S = \operatorname{diag}(Q,R) \end{aligned} \tag{18}$$

It then follows that $A_g - B_g K$ is a stable matrix and (x^*, u^*) defined by

$$\begin{aligned} &\dot{v^*} = (A_g - B_g K)v^* \quad v^*(0) = \mathcal{M}_g(Ex_0) \\ &(x^{*T}, u^{*T})^T = (C_g - D_g K)v^* \end{aligned} \tag{19}$$

is a solution of the infinite horizon optimal control problem (Problem 2) and

$$J(x^*, u^*) = (\mathcal{M}_g Ex_0)^T P \mathcal{M}_g Ex_0\,. \tag{20}$$

PROOF. [Proof of Theorem 5]

(i) $\implies$ (ii) If (x^*, u^*) is a solution of the infinite horizon optimal control problem, then by Theorem 3, there exists an input $g^* \in L^2_{loc}(0,+\infty)$ such that $\mathscr{J}(\mathcal{M}(Ex_0), g^*) < +\infty$. Let $v_0 = \mathcal{M}(Ex_0)$. We claim that if $\mathscr{J}(v_0, g^*) < +\infty$, then $\lim_{t\to\infty} v^*(t) = 0$ for the state trajectory v^* of $\mathscr{S}$ which corresponds to the input g^* and starts from v_0. The latter is equivalent to $v^*(0) = v_0 = \mathcal{M}(Ex) \in \mathcal{V}_g$. Let us prove that $\lim_{t\to\infty} v^*(t) = 0$. To this end, notice that $\mathscr{J}(v_0, g^*) < +\infty$ implies $\int_0^\infty \nu(t)^T \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \nu(t)dt < +\infty$, and as $\begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}$ is positive definite, it follows that $\mu\int_0^\infty \nu^T(t)\nu(t)dt < \int_0^\infty \nu(t)^T \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \nu(t)dt < +\infty$ for some $\mu > 0$ and so $\nu \in L_2(0,+\infty)$. Consider the decomposition $\nu(t) = (x^T(t), u^T(t))^T$, where $x(t) \in \mathbb{R}^n$. By Theorem 1 it follows that $\mathcal{M}(Ex(t)) = v^*(t)$ and hence $v^* \in L^2(0,+\infty)$. As g^* is a linear function of x, u and v (see Theorem 1) it follows that $g^* \in L_2(0,+\infty)$. Recalling that $\dot{v}^*(t) = A_l v^*(t) + B_l g^*(t)$ we write: $v^*(t) = v^*(\tau) + A_l\int_\tau^t v^*(s)ds + B_l\int_\tau^t g^*(s)ds$ for $\tau < t$. Since $\int_\tau^t \|v^*(s)\|ds \le (t-\tau)^{\frac{1}{2}}\|v^*\|_{L^2(0,+\infty)}$ where $\|v^*\|^2_{L^2(0,+\infty)} := \int_0^\infty \|v^*(s)\|^2_{\mathbb{R}^{\hat{n}}}ds$, it follows that: $\|v^*(t) - v^*(\tau)\|_{\mathbb{R}^{\hat{n}}} \le (t-\tau)^{\frac{1}{2}}(\|A_l\|\|v\|_{L^2(0,+\infty)} + \|B_l\|\|g^*\|_{L^2(0,+\infty)})$. Hence v^* is uniformly continous. This and $v^* \in L_2(0,+\infty)$ together with Barbalat's lemma imply that $\lim_{t\to\infty} v^*(t) = 0$. Hence the optimal state trajectory v^* converges to zero, i.e. g^* is a stabilizing control for v_0, and thus $v_0 = \mathcal{M}(Ex_0) \in \mathcal{V}_g$.

(ii) $\implies$ (i) Assume now that $v_0 = \mathcal{M}(Ex_0) \in \mathcal{V}_g$. Let us recall the construction of the LTI $\mathscr{S}_g$. It then follows that $T(\mathcal{V}_g) = \operatorname{im}\begin{bmatrix} I_l & 0 \\ 0 & 0 \end{bmatrix}$, where $l = \dim \mathcal{V}_g$. Define the map $\Pi = \begin{bmatrix} I_l & 0 \end{bmatrix} T$. It then follows that for any $v_0 \in V_g$, and for any $g \in L^2_{loc}(I)$, $(x^T, u^T)^T$ is the output of $\mathscr{S}$ and $t \mapsto v(t)$ is the state trajectory of $\mathscr{S}$ starting from the initial state v_0 and driven by input g, if and only if $(x^T, u^T)^T$ is the output of $\mathscr{S}_g$ and $t \mapsto \Pi(v)(t)$ is the state trajectory of $\mathscr{S}_g$ starting from $\Pi(v_0)$ and driven by the input g. For any initial state $\hat{v}_0$ of $\mathscr{S}_g$ define now the cost function $\mathscr{I}(\hat{v}_0, g, t_1)$ as

$$\begin{aligned} \mathscr{I}(\hat{v}_0, g, t_1) &= v^T(t_1)(E\hat{C}_s)^T Q_0 E\hat{C}_s v(t_1) \\ &\quad + \int_0^{t_1} \nu^T(t) S \nu(t) dt \\ \dot{v} &= A_g v + B_g g \text{ and } v(0) = \hat{v}_0 \\ \nu &= C_g v + D_g g, \quad S = \operatorname{diag}(Q,R) \end{aligned}$$

where $\hat{C}_s$ is such that $C_g = \begin{bmatrix} \hat{C}_s^T, \hat{C}_{inp}^T \end{bmatrix}^T$, such that $\hat{C}_s$ has n rows. Notice that from the definition of C_g it follows that $C_g = C_l\Pi^+$ (notice that $\Pi^+ = T^{-1}\begin{bmatrix} I_l \\ 0 \end{bmatrix}$) and hence $\hat{C}_s = C_s\Pi^+$. Define now $\mathscr{I}(\hat{v}_0, g) = \limsup_{t_1\to\infty} \mathscr{I}(\hat{v}_0, g, t_1)$. It is not hard to see that:

$$\begin{aligned} \mathscr{I}(\Pi(v_0), g, t_1) &= \mathscr{J}(v_0, g, t_1) \\ \mathscr{I}(\Pi(v_0), g) &= \mathscr{J}(v_0, g) \end{aligned} \tag{21}$$

for any initial state v_0 of $\mathscr{S}$ such that $v_0 \in \mathcal{V}_g$.

Consider now the problem of minimizing $\lim_{t_1 \to +\infty} \mathscr{I}(\hat{v}_0, g, t_1)$. Let us apply the feedback transformation $g = \hat{F}v + Uw$ to $\mathscr{S}_g = (A_g, B_g, C_g, D_g)$ with $U = (D_g^T S D_g)^{-1/2}$ and $\hat{F} = -(D_g^T S D_g)^{-1} D_g^T S C_g$, as described in [22, Section 10.5, eq. (10.32)]. Consider the linear system

$$\dot{v} = (A_g + B_g\hat{F})v + B_g U w \text{ and } v(0) = \hat{v}_0 \tag{22}$$

For any $w \in L^2_{loc}(I)$, where $I = [0, t_1]$ of $I = [0, +\infty)$, the state trajectory v of (22) equals the state trajectory of $\mathscr{S}_g$ for the input $g = \hat{F}v + Uw$ and initial state $\hat{v}_0$. Moreover, all inputs g of $\mathscr{S}_g$ can be represented in such a way. Define now

$$\widehat{\mathscr{I}}(\hat{v}_0, w, t) = v^T(t)(E\hat{C}_s)^T Q_0 E\hat{C}_s v(t) + \\ + \int_0^t (v^T(t)(C_g + D_g\hat{F})^T S (C_g + D_g\hat{F})v(t) + w^T(t)w(t))dt,$$

where v is a solution of (17). Due to the construction of $\hat{F}$ and U, $\|S^{1/2}(C_g + D_g\hat{F})v(t) + D_g U w(t)\|^2 = v^T(t)(C_g + D_g\hat{F})^T S(C_g + D_g\hat{F})v(t) + w^T(t)w(t)$. Using these remarks, it is then easy to see that $\mathscr{I}(\hat{v}_0, g, t) = \widehat{\mathscr{I}}(\hat{v}_0, w, t)$ for $g = \hat{F}v + Uw$.

Consider now the problem of minimizing $\lim_{t\to\infty} \widehat{\mathscr{I}}(\hat{v}_0, w, t)$. We apply [11, Theorem 3.7]. To this end, notice that $(A_g + B_g\hat{F}, B_g U)$ is stabilizable and $(S^{1/2}(C_g + D_g\hat{F}), A_g + B_g\hat{F})$ is observable. Indeed, it is easy to see that stabilizability of (A_g, B_g) implies that of $(A_g + B_g\hat{F}, B_g U)$. Observability of $(S^{1/2}(C_g + D_g\hat{F}), A_g + B_g\hat{F})$ can be derived as follows. Recall from Theorem 1 that EC_s is of full column rank and $ED_s = 0$. Let $\hat{D}_s$ be the sub-matrix of D_g formed by its first n rows. Hence, $E\hat{D}_s = ED_s = 0$. Moreover, since $E\hat{C}_s$ equals the matrix representation of the restriction of the map $E(C_s + D_s\hat{F}) = EC_s$ to $\mathcal{V}_g$, it follows that $E\hat{C}_s$ is of full column rank, if EC_s is injective. The latter is the case according to Theorem 1. Hence, $E(\hat{C}_s + \hat{D}_s\hat{F}) = E\hat{C}_s$ is of full column rank, and thus the pair $(\hat{C}_s + \hat{D}_s\hat{F}, A_g + B_g\hat{F})$ is observable. Let us now return to the minimization problem. Notice that (18) is equivalent to the algebraic Riccati equation described in [11, Theorem 3.7] for the problem of minimizing $\lim_{t\to\infty} \widehat{\mathscr{I}}(\hat{v}_0, w, t)$. Hence, by [11, Theorem 3.7], (18) has a unique positive definite solution P, and $A_g + B_g\hat{F} - B_g U U^T B_g^T P = A_g - B_g K$ is a stable matrix. From [11, Theorem 3.7], there exists w^* such that $\lim_{t\to\infty} \widehat{\mathscr{I}}(\hat{v}_0, w^*, t)$ is minimal and $\hat{v}_0^T P \hat{v}_0 = \lim_{t\to\infty} \widehat{\mathscr{I}}(\hat{v}_0, w^*, t)$. On the other hand, [11, Theorem 3.7] also implies that $\hat{v}_0^T P \hat{v}_0 = \lim_{t_1\to\infty} \inf_{w \in L^2(0,t_1)} \widehat{\mathscr{I}}(\hat{v}_0, w, t_1)$. Hence, $g^* = \hat{F}v^* + Uw^*$ satisfies

$$\mathscr{I}(\hat{v}_0, g^*) = \hat{v}_0^T P \hat{v}_0 = \lim_{t\to\infty} \inf_{g \in L^2(0,t_1)} \mathscr{I}(\hat{v}_0, g, t),$$

where $\dot{v}^* = (A_g + B_g\hat{F})v^* + B_g U w^*$, $v^*(0) = \hat{v}_0$. A routine computation reveals that (v^*, g^*) satisfies

$$\dot{v}^* = A_g v^* + B_g g^* \text{ and } g^* = -Kv^* \text{ and } v^*(0) = \hat{v}_0.$$

From Theorem 3 and Part 4b of Theorem 1 it then follows that $(x^{*T}, u^{*T})^T = C_g v^* + D_g g^*$ is the solution of the infinite horizon optimal control problem and that (x^*, u^*) satisfies (19) and (20).

(i) $\implies$ **(iii)** If $(x^*, u^*) \in \mathcal{D}_{x_0}(\infty)$ is a solution of the infinite horizon optimal control problem, then, by definition, $+\infty > J(x^*, u^*) = \limsup_{t_1\to\infty} \inf_{(x,u)\in\mathcal{D}_{x_0}([0,t_1])} J(x, u, t_1)$.

(iii) $\implies$ **(ii)** From (11) it follows that the condition of **(iii)** implies that there exists $M > 0$ such that for all $t_1 > 0$,

$$\inf_{g \in L^2(0,t_1)} \mathscr{J}(\mathcal{M}(Ex_0), g, t_1) \le M. \tag{23}$$

Recall that $S = \operatorname{diag}(Q, R)$ and for each $g \in L^2(I)$, $[0, t_1] \subseteq I$, and initial state v_0, define

$$\mathscr{I}(v_0, g, t_1) = \int_0^{t_1} (C_l v(t) + D_l g(t))^T S (C_l v(t) + D_l g(t)) dt \\ \dot{v} = A_l v + B_l g, \;\; v(0) = v_0$$

It then follows that $\mathscr{I}(v_0, g, t_1) \le \mathscr{J}(v_0, g, t_1)$ for any $t_1 > 0$ and $\mathscr{I}(v_0, g, t_1)$ is non-decreasing in t_1. Hence, (23) implies that

$$\forall t_1 \in (0, +\infty) : \inf_{g \in L^2(0,t_1)} \mathscr{I}(v_0, g, t_1) < M.$$

From classical LTI theory [11, 22] it follows that if H is the unique symmetric, positive semi-definite solution of the Riccati equation

$$\dot{H}(t) = A_l^T H(t) + H(t)A_l - K^T(t)(D_l^T S D_l)K(t) + C_l^T S C_l \\ \text{and } H(0) = 0 \\ K(t) = (D_l^T S D_l)^{-1}(B_l^T H(t) + D_l^T S C_l) \tag{24}$$

then $v_0^T H(t_1) v_0 = \inf_{g \in L_2(0,t_1)} \mathscr{I}(v_0, g, t_1)$. The latter may be easily seen applying the state feedback transformation $g = \hat{F}v + Uw$ with F, U defined as in the proof of Theorem 4 and solve the resulting standard LQ control problem for the transformed system. It then follows that (24) is the differential Riccati equation which is associated with this problem. Note that the matrix $\dot{H}(t)$

is symmetric and positive semi-definite, since $b^T H(t)b$ is monotonically non-decreasing for all b. Define the set

$$V = \{v_0 \mid \sup_{t_1 \in (0,+\infty)} v_0^T H(t_1) v_0 < +\infty\}. \tag{25}$$

By assumption **(iii)** it follows that $\mathcal{M}(Ex_0) \in V$. From [22, Theorem 10.13] it follows that [1] $\mathcal{V}_g \subseteq V$. It is also easy to see that V is a linear space. We will show that $V = \mathcal{V}_g$, from which $\mathcal{M}(Ex_0) \in \mathcal{V}_g$ follows.

First, we will argue that V is invariant with respect to A_l. To this end, consider $v_0 \in V$ and set $v_1 = e^{-A_l t} v_0$. For any t_1 and any $g \in L^2(0,t_1)$, define $\hat{g} \in L^2_{loc}(0, t_1+t)$ as $\hat{g}(s) = 0$, $s \le t$ and $\hat{g}(s) = g(s-t)$ if $s > t$. Consider $\dot{v} = A_l v + B_l \hat{g}$, $v(0) = v_1$. It then follows $v(t) = v_0$ and hence

$$\begin{aligned} &\mathscr{J}(v_1, \hat{g}, t+t_1) = \\ &\mathscr{J}(v_0, g, t_1) + \int_0^t (C_l e^{A_l s} v_1)^T S (C_l e^{A_l s} v_1) ds \end{aligned} \tag{26}$$

Since $v_0 \in V$, it then follows that there exists $\Gamma > 0$ such that for any t_1 there exists $g \in L^2(0,t_1)$ such that $v_0^T H(t_1) v_0 = \mathscr{J}(v_0, g, t_1) \le \Gamma$. Hence, from (26) it follows that $v_1^T H(t_1 + t) v_1 = \inf_{\bar{g} \in L^2(0,t_1+t)} \mathscr{J}(v_1, \bar{g}, t+t_1) \le \Gamma + \int_0^t (C_l e^{A_l s} v_1)^T S (C_l e^{A_l s} v_1) ds < +\infty$, and hence

$$\sup_{t_1 \in (0,+\infty)} v_1^T H(t_1) v_1 \le \sup_{t_1 \in (0,+\infty)} v_1^T H(t_1 + t) v_1 < +\infty.$$

In the last step we used that $v_1^T H(t+t_1) v_1 \ge v^T H(t_1) v$ for all $t, t_1 \in [0,+\infty)$. Hence, $v_1 = e^{-A_l t} v_0 \in V$. Since V is a linear space, and t is arbitrary, it then follows that $A_l v_0 = -\frac{d}{dt} e^{-A_l t} v_0|_{t=0} \in V$. That is, V is A_l invariant.

Notice that the controllability subspace of $\mathscr{S}$ is contained in $\mathcal{V}_g \subseteq V$ and that $\mathrm{im} B_l$ is contained in the controllability subspace $\mathscr{S}$. Hence, $\mathrm{im} B_l \subseteq V$. Notice that for any $b \in V$, the function $b^T H(t) b$ is monotonically non-decreasing in t and it is bounded, hence $\lim_{t\to\infty} b^T H(t) b$ exists and it is finite. Notice for any $b_1, b_2 \in V$, $(b_1 + b_2)^T H(t)(b_1 + b_2) = b_1^T H(t) b_1 + 2 b_1^T H(t) b_2 + b_2^T H(t) b_2$ and as $b_1 + b_2 \in V$, the limit on both sides exists and so the limit $\lim_{t\to\infty} b_1^T H(t) b_2$ exists. Consider now a basis $b_1, \dots, b_r$ of V and for any t define $\hat{H}_{i,j}(t) = b_i^T H(t) b_j$. It then follows that the matrix $\hat{H}(t) = (H_{i,j}(t))_{i,j=1,\dots,r}$ is positive semi-definite, symmetric and there exists a positive semi-definite matrix $\hat{H}_+$ such that $\hat{H}_+ = \lim_{t\to\infty} \hat{H}(t)$. From (24) and $A_l V \subseteq V$ it follows that $A_l b_i \in V$ for all $i = 1, \dots, r$ and hence $\lim_{t\to t} b_i^T H(t) A_l b_j$, $\lim_{t\to t} b_i^T A_l^T H(t) b_j$ exist for all $i,j = 1,\dots,r$. From $\mathrm{im} B_l \subseteq V$ and the fact that for any $x, z \in V$, the limit $\lim_{t\to\infty} x^T H(t) z$ exists, it follows that the limits $\lim_{t\to\infty} B_l^T H(t) x = \lim_{t\to\infty} x^T H(t) B_l$ exist for all $x \in V$. Applying this remark to $x = b_i$ and $x = b_j$, it follows that $\lim_{t\to\infty} b_i^T K(t)^T (D_l^T S D_l) K(t) b_j$ exists. Hence, for any $i,j = 1,\dots,r$, the limit of $\dot{\hat{H}}_{i,j}(t) = b_i^T \dot{H}(t) b_j$ exists as $t \to \infty$ and hence the limit $\lim_{t\to\infty} \dot{\hat{H}}(t) =: Z$ exists. Moreover, since $\dot{H}(t)$ is symmetric and positive semi-definite, it follows that Z is symmetric and positive semi-definite.

We claim that Z is zero. To this end it is sufficient to show that $\lim_{t\to\infty} b^T \dot{H}(t) b = 0$ for any $b \in V$. Indeed, from this it follows that $Z_{i,j} = \lim_{t\to\infty} 0.5((b_i + b_j)^T \dot{H}(t)(b_i + b_j) - b_i^T \dot{H}(t) b_i - b_j^T \dot{H}(t) b_j) = 0$ for any $i,j = 1,\dots,r$. Now, fix $b \in V$ and assume that $c = \lim_{t\to\infty} b^T \dot{H}(t) b \neq 0$. Set now $h(t) = b^T H(t) b$. It then follows that $\dot{h}(t) = b^T \dot{H}(t) b$ and thus there exists $T > 0$ such that for all $t > T$, $\dot{h}(t) > \frac{c}{2} > 0$ (note that $\dot{H}(t)$ is positive semi-definite). Hence, $h(t) = h(0) + \int_0^t \dot{h}(s) ds > \int_T^t \dot{h}(s) ds > (t-T)\frac{c}{2}$. Hence, $h(t)$ is not bounded, which contradicts to the assumption that $b \in V$.

Hence, $Z = 0$ and thus, it follows that $\hat{H}_+$ satisfies the algebraic Riccati equation

$$\begin{aligned} &0 = \hat{A}_l \hat{H}_+ + \hat{H}_+ \hat{A}_l - K^T (D_l S D_l) K + \hat{C}_l^T S \hat{C}_l \\ &K = (D_l^T S D_l)^{-1} (\hat{B}_l^T \hat{H}_+ + D_l^T S \hat{C}_l) \end{aligned} \tag{27}$$

where $\hat{A}_l, \hat{B}_l, \hat{C}_l$ are defined as follows: $\hat{A}_l$ and $\hat{C}_l$ are the matrix representations of the linear maps A_l and C_l restricted to V, $\hat{B}_l$ is the matrix representation of the map $\mathbb{R}^k \ni g \mapsto B_l g \in V$ in the basis $b_1, \dots, b_r$ of V chosen as above. Note that for $\hat{A}_l$ and $\hat{B}_l$ to be well defined, we had to use the facts $\mathrm{im} B_l \subseteq V$ and $A_l V \subseteq V$. Notice that D_l is injective as a linear map and recall from Remark 3 that $C_l p + D_l g = 0$ implies that $p = 0$, $g = 0$ and hence the largest output nulling subspace $\mathcal{V}(\hat{\Sigma})$ of the linear system $\hat{\Sigma} = (\hat{A}_l, \hat{B}_l, \hat{C}_l, D_l)$ is zero. Then from [22, Theorem 10.19], $\mathcal{V}(\hat{\Sigma}) = 0$ and (27) it follows that $\hat{\Sigma}$ is stabilizable. Since Σ is just the restriction of $\mathscr{S}$ to V, it then follows that every state from V is stabilizable and hence $V = \mathcal{V}_g$.

Remark 8 (Infinite-horizon case) *The proof of Theorem 5 implies that in the formulation of associated linear control problem, we can replace* lim sup *by* lim.

[1] Indeed, recall from the system $\mathscr{S}_g = (A_g, B_g, C_g, D_g)$ and the map Π. Recall that $\mathscr{S}_g$ is stabilizable and hence by [22, Theorem 10.19] for every $v_0 \in \mathcal{V}_g$ there exists an input g such that $\int_0^\infty ((C_g v(s) + D_g g(s))^T S (C_g v(s) + D_g g(s)) ds < +\infty$ for $\dot{v} = A_g v + B_g g$, $v(0) = \Pi(v_0)$. Consider the state trajectory $r(t)$ where $\dot{r} = A_l r + B_l g$, $r(0) = v_0$. It then follows that $\Pi(r) = v$ and hence $C_l r + D_l g = C_g v + D_g g$. Therefore, $v_0^T H(t_1) v_0 \le \mathscr{J}(v_0, g, t_1) \le \int_0^\infty ((C_g v(s) + D_g g(s))^T S (C_g v(s) + D_g g(s)) ds < +\infty$.

Remark 9 (Computation) *The existence of solution for Problem 1 and Problem 2 and its computation depend only on the matrices (E, A, B, Q, R, Q_0). Indeed, a linear system $\mathscr{S}$ associated with (E, A, B) can be computed from (E, A, B), and the solution of the associated LQ problem can be computed using $\mathscr{S}$ and the matrices Q, Q_0, R. Notice that the only condition for the existence of a solution is the stabilizability from x_0.*

4 Conclusions

We have presented the solution to infinite horizon linear quadratic control problem for generic linear DAEs. Specifically, for any tuple (E, A, B) and initial condition x_0 we proved that the infinite horizon control problem is solvable if and only if the DAE Σ is stabilizable from x_0. We also proposed an intuitive and computationally feasible way to check the stabilizability and construct the optimal control which makes the proposed framework attractive for applications.

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PROOF. [Proof of Lemma 1] Consider two associated linear systems $\mathscr{S}_i = (A_{i,l}, B_{i,l}, C_{i,l}, D_{i,l})$, $i = 1, 2$ each of which correspond to the choosing $T = T_i, S = S_i, \widetilde{F} = F_i, L = L_i, \mathscr{R} = \mathscr{R}_i$, $i = 1,2$ in the proof of Theorem 1. Consider the maps $\mathcal{M}_i$, $i = 1,2$ which correspond to $\mathscr{S}_i$, $i = 1,2$. Consider the decomposition

$$S_i E T_i = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

$$S_i \hat{A} T_i = \begin{bmatrix} A_i & A_{12,i} \\ A_{21,i} & A_{22,i} \end{bmatrix}, \quad S_i \hat{B} = \begin{bmatrix} B_{1,i} \\ B_{2,i} \end{bmatrix},$$

$$G_i = \begin{bmatrix} A_{12,i}, & B_{1,i} \end{bmatrix}, \quad \widetilde{C}_i = A_{21,i} \text{ and } \widetilde{D}_i = \begin{bmatrix} A_{22,i}, & B_{2,i} \end{bmatrix}$$

and consider the linear systems

$$\mathscr{S}_i \begin{cases} \dot{p}_i = A_i p_i + G_i q_i \\ z_i = \widetilde{C}_i p_1 + \widetilde{D}_i q_i \end{cases}$$

for $i = 1,2$. Denote by $\mathcal{V}_i = \mathcal{V}(\mathscr{S}_i)$ the set of weakly unobservable states of $\mathscr{S}_i$, $i = 1,2$. Denote by $\mathcal{F}(\mathcal{V}_i)$, $i = 1,2$, the set of all state feedback matrices $F \in \mathbb{R}^{(n+m)\times r}$ such that $(A_i + G_i F)\mathcal{V}_i \subseteq \mathcal{V}_i$, $(\widetilde{C}_i + \widetilde{D}_i F)\mathcal{V}_i = 0$. Pick $F_i \in \mathcal{F}(\mathcal{V}_i)$, $i = 1,2$ and pick full column rank matrices L_i, $i = 1,2$ such that $\text{im} L_i = G_i^{-1}(\mathcal{V}_i) \cap \ker \widetilde{D}_i$. In order to prove the lemma, it is enough to show that $\text{Rank} L_1 = \text{Rank} L_2 = k$, and there exist invertible linear maps $X \in \mathbb{R}^{r\times r}$, $U \in \mathbb{R}^{k\times k}$, and a matrix $F \in \mathbb{R}^{k\times r}$ such that

$$X(\mathcal{V}_1) = \mathcal{V}_2 \tag{.1a}$$

$$(A_1 + G_1 F_1 + G_1 L_1 F)\mathcal{V}_1 \subseteq \mathcal{V}_1 \tag{.1b}$$

$\forall x \in \mathcal{V}_1$:

$$X(A_1 + G_1 F_1 + G_1 L_1 F)x = (A_2 + G_2 F_2)Xx \tag{.1c}$$

$$X G_1 L_1 U = G_2 L_2 \tag{.1d}$$

$$\begin{bmatrix} T_1 & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} 0_{r\times k} \\ L_1 U \end{bmatrix} = \begin{bmatrix} T_2 & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} 0_{r\times k} \\ L_2 \end{bmatrix} \tag{.1e}$$

$\forall x \in \mathcal{V}_1$:

$$\begin{bmatrix} T_1 & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} I_r \\ (F_1 + L_1 F) \end{bmatrix} x = \begin{bmatrix} T_2 & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} I_r \\ F_2 \end{bmatrix} Xx, \tag{.1f}$$

where $0_{r\times k}$ denotes the $r \times k$ matrix with all zero entries. Note that (.1) implies that $\hat{n} = \dim \mathcal{V}_1 = \dim \mathcal{V}_2$. Since $A_{l,i} = \begin{bmatrix} I_{\hat{n}} & 0 \end{bmatrix} \mathscr{R}_i (A_i + G F_i) \mathscr{R}_i^{-1} \begin{bmatrix} I_{\hat{n}} \\ 0 \end{bmatrix}$, $C_{l,i} = \begin{bmatrix} T_i & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} I_r \\ F_i \end{bmatrix} \mathscr{R}_i^{-1} \begin{bmatrix} I_{\hat{n}} \\ 0 \end{bmatrix}$, $B_{l,i} = \begin{bmatrix} I_{\hat{n}} & 0 \end{bmatrix} \mathscr{R}_i G_i L_i$, $D_{l,i} = \begin{bmatrix} T_i & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} 0 \\ L_i \end{bmatrix}$, and $\text{im} G_i L_i \subseteq \mathcal{V}_i$, $\mathscr{R}_i(\mathcal{V}_i) = \text{im} \begin{bmatrix} I_{\hat{n}} & 0 \\ 0 & 0 \end{bmatrix}$, if (X, F, U) satisfy (.1), it then follows that $(\begin{bmatrix} I_{\hat{n}} & 0 \end{bmatrix} \mathscr{R}_2 X \mathscr{R}_1^{-1} \begin{bmatrix} I_{\hat{n}} \\ 0 \end{bmatrix}, F\mathscr{R}_1^{-1} \begin{bmatrix} I_{\hat{n}} \\ 0 \end{bmatrix}, U)$ is a feedback equivalence between $\mathscr{S}_1$ and $\mathscr{S}_2$. Finally, $(\hat{T}, \hat{F}, \hat{U})$ is a feedback equivalence from $\mathscr{S}_1$ to $\mathscr{S}_2$, and $C_{s,1}, C_{s,2}, D_{s,1}, D_{s,2}$ denote the first n rows of $C_{l,1}, C_{l,2}, D_{l,1}, D_{l,2}$ respectively, then $EC_{s,2} = E(C_{s,1} + D_{s,1}\hat{F})\hat{T}^{-1} = EC_{s,1}\hat{T}^{-1}$, since $ED_{s,1} = 0$. Hence, $\mathcal{M}_2 = (EC_{s,2})^+ = (EC_{s,1}\hat{T}^{-1})^+ = \hat{T}(EC_{s,1})^+ = \hat{T}\mathcal{M}_1$.

In order to find the matrices F, U, X, notice that

$$T_2^{-1} T_1 = \begin{bmatrix} R_{11} & 0 \\ R_{21} & R_{22} \end{bmatrix}, \quad S_2 S_1^{-1} = \begin{bmatrix} H_{11} & H_{21} \\ 0 & H_{22} \end{bmatrix}$$
$$R_{11} = H_{11}$$

where $H_{11} \in \mathbb{R}^{r\times r}$, $H_{22} \in \mathbb{R}^{(c-r)\times(c-r)}$, $H_{12} \in \mathbb{R}^{r\times(c-r)}$, $R_{11} \in \mathbb{R}^{r\times r}$, $R_{22} \in \mathbb{R}^{(n-r)\times(n-r)}$, $R_{21} \in \mathbb{R}^{(n-r)\times r}$. We show the equality for $T_2^{-1}T_1$, the one for $S_2 S_1^{-1}$ can be shown analogously (see [16]). Assume $T_2^{-1} T_1 \begin{bmatrix} 0 \\ q \end{bmatrix} = \begin{bmatrix} \bar{p} \\ \bar{q} \end{bmatrix}$ for some $q, \bar{q} \in \mathbb{R}^{n-r}$, $\bar{p} \in \mathbb{R}^r$. Then $\begin{bmatrix} \bar{p} \\ 0 \end{bmatrix} = S_2 E T_2 T_2^{-1} T_1 \begin{bmatrix} 0 \\ q \end{bmatrix} = S_2 S_1^{-1} S_1 E T_1 \begin{bmatrix} 0 \\ q \end{bmatrix} = 0$. $R_{11} = H_{11}$ follows from $\begin{bmatrix} H_{11} & 0 \\ 0 & 0 \end{bmatrix} = S_2 S_1^{-1}(S_1 E T_1) = (S_2 E T_2) T_2^{-1} T_1 = \begin{bmatrix} R_{11} & 0 \\ 0 & 0 \end{bmatrix}$.

From $S_2 \hat{A} T_2 = S_2 S_1^{-1} S_1 \hat{A} T_1 (T_2^{-1} T_1)^{-1}$ it and $S_2 \hat{B} = S_2 S_1^{-1} S_1 \hat{B}$ follows that

$$\begin{aligned} A_2 &= R_{11}(A_1 + G_1 \hat{F} + \hat{G}\widetilde{C}_1 + \hat{G}\widetilde{D}_1 \hat{F}) R_{11}^{-1} \\ G_2 &= R_{11}(G_1 + \hat{G}\widetilde{D}_1)\hat{U} \\ \widetilde{D}_2 &= \hat{V}\widetilde{D}_1 \hat{U} \text{ and } \widetilde{C}_2 = \hat{V}(\widetilde{C}_1 + \widetilde{D}_1 \hat{F}) R_{11}^{-1} \end{aligned} \tag{.2}$$

where $\hat{F} = \begin{bmatrix} -R_{22}^{-1} R_{12} \\ 0 \end{bmatrix}$, $\hat{G} = R_{11}^{-1} H_{12}$, $\hat{U} = \begin{bmatrix} R_{22}^{-1} & 0 \\ 0 & I_m \end{bmatrix}$, and $\hat{V} = H_{22}$.

We then claim that the following choice of matrices

$$\begin{aligned} X &= R_{11} \text{ and } U = L_1^+ \hat{U} L_2 \\ F &= L_1^+(\hat{F} + \hat{U} F_2 R_{11} - F_1) \end{aligned} \tag{.3}$$

satisfies (.1). The proof of (.1a) – (.1f) is routine, the interested reader can find the details in [16].