Minimax Observers for Linear DAEs

Sergiy Zhuk and Mihaly Petreczky

Abstract

In this note we construct finite and infinite horizon minimax observers for a linear stationary DAE with deterministic, unknown, but bounded noise. By using generalized Kalman duality and geometric control we prove that the finite (infinite) horizon observer exists if and only if the DAE is observable (detectable). Remarkably, the regularity for the DAE is not required.

I. Introduction

Consider a linear Differential-Algebraic Equation (DAE) in the following form:

$$\begin{aligned} &\frac{d(Fx)}{dt} = Ax(t) + f(t), \quad Fx(t_0) = x_0\,, \\ &y(t) = Hx(t) + \eta(t) \end{aligned} \tag{1}$$

where $F, A \in \mathbb{R}^{m\times n}$ and $H \in \mathbb{R}^{p\times n}$ are given matrices, and (x_0, f, η) belong to the set

$$\mathscr{E} = \{(x_0, f, \eta) \mid x_0^T Q_0 x_0 + \int_0^\infty f^T(s)Qf(s) + \eta^T(s)R\eta(s)ds \le 1\}\,.$$

The ellipsoid $\mathscr{E}$ can be viewed as a set describing admissible noises and initial state of the system. We aim at estimating a linear function $\ell^T Fx(t)$, $\ell \in \mathbb{R}^m$ of the state vector $x(t)$ of (1). The desired estimate $\mathcal{O}_U(t, y)$ should be linear in y, that is $\mathcal{O}_U(t, y) := \int_0^t y^T(s)U(t, s)ds$. Our goal is to find an estimate $\widehat{\mathcal{O}}_{\widehat{U}}(t, y)$ such that: the worst-case estimation error $\sigma := \limsup_{t\to\infty} \sup_{(x_0,f,\eta)\in\mathscr{E}} (\ell^T Fx(t) - \widehat{\mathcal{O}}_{\widehat{U}}(t, y))^2$ is minimal, and $\widehat{\mathcal{O}}_{\widehat{U}}(t, y) = \int_0^t y^T(s)\widehat{U}(t, s)ds$ may be represented as an output of a stable LTI system whose input is y. We will refer to $\mathcal{O}_U(t, y)$ and $\widehat{\mathcal{O}}_{\widehat{U}}(t, y)$ (and $U, \widehat{U}$) as *observer* and *minimax observer* respectively.

S. Zhuk is with IBM Research - Ireland, Damastown, Dublin 15, Ireland (e-mail: sergiy.zhuk@ie.ibm.com)

M. Petreczky is with Ecole des Mines de Douai, Dept. of Comp. Sci. & Autom. Control, 59508 Douai, France (e-mail: mihaly.petreczky@mines-douai.fr)

Corresponding author: M. Petreczky

1) Contribution of the paper: We show that the minimax observer design problem for (1) is dual to an infinite horizon LQ control problem for the adjoint DAE. This result generalizes Kalman duality and allows us to introduce a notion of ℓ-detectability for generic DAEs. By using duality we prove that the minimax observer exists if and only if (1) is ℓ-detectable.

2) Motivation: The minimax approach is one of many classical ways (see for instance [1]–[4]) to pose a state estimation problem. Apart from theoretical reasons, our interest in designing the minimax observer for DAEs is motivated by applications. For example, the minimax projection method proposed in [5], [6] allows one to project a partial differential equation onto a finite dimensional subspace and derive a DAE for the projection coefficients. The resulting DAE is subject to the projection error which is deterministic, unknown but bounded. Thus, the minimax observer may be applied to estimate the projection coefficients. Besides, DAEs are applied in robotics [7] and modelling [8].

3) Related work: To the best of our knowledge, the main result of this paper is new. A preliminary version of this paper appeared in [9]. With respect to [9] the main difference is that we introduced ℓ-detectability, proved that it is equivalent to the existence of the minimax observer, and included detailed proofs for the duality theorems. The case of finite horizon minimax observer was addressed in [10]. Unlike [10] we include a detailed proof for the duality theorem and solve the dual control problem by means of geometric control methods which contain a solution proposed in [10] as a special case. A sub-optimal infinite horizon minimax observer for non-stationary DAEs with trivial initial condition $Fx(0) = 0$ was presented in [11]. In contrast to [11], our duality theorems hold for DAEs with uncertain $Fx(0) = x_0$ and lead to non-trivial optimal infinite horizon observer $\widehat{U}$ which can be realized by a stable LTI. Although the literature on state estimation for DAEs is vast, most of the papers concentrate on regular DAEs, see [12] for an overview. In [13] the problem of parameter estimation for DAEs is considered, provided the noise f is smooth. The latter assumption is required in order to use the Weistrass canonical form. The papers [14], [15] consider the problem of finding observers such that the H_∞ norm of the error system is smaller than a certain upper bound. In contrast, minimax observers considered in this paper minimize a different error measure, namely, the worst-case estimation error σ. Moreover, unlike [14], we allow for non-regular DAEs, and unlike [15], we do not require impulsive observability. Furthermore, in [15] solvability of an LMI was required in order to ensure existence of an observer. In contrast, in our setting detectability is necessary and sufficient for existence of the minimax observer. We elaborate on the relationship to [14], [15] in Remark 4. In [16] the problem of unknown input observer design for DAEs was considered. Unlike in [16], we assume that the unknown input (i.e. noise) is bounded, and we aim at recovering only a part of the state. For this reason, our results are not directly comparable to [16].

4) Outline of the paper: Section II describes the mathematical problem statement, sections III-IV present main results of the paper.

5) Notation: $\mathcal{I}_n$ denotes the $n \times n$ identity matrix; For an $n \times n$ matrix S, $S > 0$ means $x^T S x > 0$ for all $x \in \mathbb{R}^n$; F^+ denotes the pseudoinverse of a matrix F. For $I := [0, t]$, $t \leq +\infty$ let $L^2(I, \mathbb{R}^n)$ denote the space of all square-integrable functions $f : I \to \mathbb{R}^n$ and set $L^2_{loc}(I, \mathbb{R}^n) = \{f \in L^2(I^1, \mathbb{R}^n), \forall I^1 \subsetneq I, \ I^1$ is a compact interval$\}$. We will often write $L^2(0, t)$ and $L^2_{loc}(0, t)$ referring to $L^2(I, \mathbb{R}^n)$ and $L^2_{loc}(I, \mathbb{R}^n)$ respectively. $\mathscr{D}(L)$, $\mathrm{im} L$, $\ker L$ denote the domain, range and null-space of the operator L respectively and L' denotes the adjoint operator. $f|_A$ stands for the restriction of a function f onto a set A.

II. Problem statement

Consider the DAE (1). Assume that $I = [0, t_1]$, $0 < t_1 \in \mathbb{R}$ or $I = [0, \infty+)$ and $f \in L^2(I, \mathbb{R}^m)$ and $\eta \in L^2(I, \mathbb{R}^p)$. We say that $t \mapsto x(t) \in \mathbb{R}^n$ is a solution of (1) on I if $x \in L^2_{loc}(I)$, Fx is absolutely continuous and x satisfies (1) almost everywhere on I. By defining the solution of (1) in this way, we allow for state vectors x with non-differentiable parts belonging to the null-space of F. We refer to [11] for a detailed discussion.

We further assume that initial condition x_0, model error f and output noise η are unknown and belong to the given set $\mathscr{E}(t_1) := \{(x_0, f, \eta) \in \mathbb{R}^n \times L^2(I, \mathbb{R}^n) \times L^2(I, \mathbb{R}^p) : \rho(x_0, f, \eta, t_1, Q_0, Q, R) \leq 1\}$, where $Q_0, Q(t) \in \mathbb{R}^{m \times m}$, $Q_0 = Q_0^T > 0$, $Q = Q^T > 0$, $R \in \mathbb{R}^{p \times p}$, $R^T = R > 0$ and

$$\rho(x_0, f, \eta, t_1, Q_0, Q, R) := x_0^T Q_0 x_0 + \int_0^{t_1} f^T Q f + \eta^T R \eta dt \,. \tag{2}$$

Notation 1 (Solution set $\mathscr{E}\mathscr{E}(t_1)$): Denote by $\mathscr{E}\mathscr{E}(t_1)$ the set of all tuples (x_0, x, y, f, η) such that $(x, y) \in L^2([0, t_1], \mathbb{R}^n) \times L^2([0, t_1], \mathbb{R}^p)$, Fx is absolutely continuous, $(x_0, f, \nu) \in \mathscr{E}(t_1)$, and (x_0, x, y, f, η) satisfies (1) almost everywhere.

Definition 1 (Finite horizon observers): Given $\ell \in \mathbb{R}^m$ and the output y of (1) on $I := [0, t_1]$, $t_1 < +\infty$ we define an observer U as a linear estimate of $\ell^T F x(t_1)$ in the form:

$$\mathcal{O}_U(t_1, y) = \int_0^{t_1} y^T(t) U(t) dt \,, \quad U \in L^2(I) \,.$$

To each observer U we assign a worst-case estimation error:

$$\sigma(U,\ell,t_1) := \sup_{(x_0,x,y,f,\eta)\in\mathscr{E}\mathscr{E}(t_1)} (\ell^T F x(t_1) - \mathcal{O}_U(t_1,y))^2 \,.$$

An observer $\widehat{U}$ is called a *minimax observer*, if $\sigma(\widehat{U},\ell,t_1) = \inf_{U\in L^2(0,t_1)} \sigma(U,\ell,t_1) < +\infty$. Intuitively, the estimation error $\sigma(U,\ell,t_1)$ can be interpreted as finding the "worst" realisation of the uncertain parameters $(x_0,f,\eta)\in\mathscr{E}(t_1)$ for a given observer U. Consequently, the minimax observer $\widehat{U}$ is the observer with the minimal worst-case error — the minimax error.

Definition 2 (Infinite horizon observers): Denote by $\mathscr{F}$ the set of all maps $U : \{(\tau,s) \mid \tau > 0, s\in[0,\tau]\} \to \mathbb{R}^p$ such that $\forall\tau>0$, the map $U(\tau,\cdot) : [0,\tau]\ni s\mapsto U(\tau,s)$ belongs to $L^2(0,\tau)$.

An element $U\in\mathscr{F}$ will be called an *infinite horizon observer*. If $I=[0,\tau]\cap\mathbb{R}$, $0<\tau\le+\infty$ and $y\in L^2_{loc}(I,\mathbb{R}^p)$, then the result of applying U to y is a function $\mathscr{O}_U(y) : I\to\mathbb{R}$, such that

$$\forall t\in I : \mathscr{O}_U(y)(t) = \mathcal{O}_{U(t,\cdot)}(t,y) = \int_0^t y^T(s)U(t,s)ds \,,$$

where $U(t,\cdot)$ is a finite horizon observer. The worst-case error for $U\in\mathscr{F}$ is defined as $\sigma(U,\ell) := \limsup_{t\to\infty}\sigma(U(t,\cdot),t,\ell)$. An infinite horizon observer $\widehat{U}$ is called a *minimax observer*, if

$$\sigma(\widehat{U},\ell) = \inf_{U\in\mathscr{F}} \sigma(U,\ell) < +\infty \,. \tag{3}$$

In fact, an infinite horizon observer $U\in\mathscr{F}$ is a collection of finite horizon observers $U(\tau,\cdot)$ associated to the time interval $I=[0,\tau]$, and infinite horizon minimax observers have the minimal asymptotic worst-case estimation error. So far we have defined observers as linear integral operators mapping observations to state estimates. For practical purposes it is desirable that the observer is represented by a stable LTI system.

Definition 3: The observer $U\in\mathscr{F}$ is said to be represented by a stable LTI system, if there exists $A_o\in\mathbb{R}^{r\times r}, B_o\in\mathbb{R}^{r\times p}, C_o\in\mathbb{R}^{1\times r}$ such that A_o is stable, and for any interval $I=[0,t_1]\cap\mathbb{R}$, $0<t_1\le+\infty$ and any $y\in L^2_{loc}(I)$, it holds that $\forall t\in I : \mathscr{O}_U(y)(t) = C_o s(t)$, where $\dot{s}(t) = A_o s(t) + B_o y(t)$, $s(0)=0$.

We are now ready to state formally the two problems addressed in this paper.

Problem 1 (Minimax observer design problem):

- **Finite horizon** Construct the finite horizon minimax observer.
- **Infinite horizon** Construct the infinite horizon minimax observer $\widehat{U}$ such that $\widehat{U}$ can be represented by a stable LTI system.

III. Dual control problem

Below we show that the problem of minimax observer design can be solved by solving an optimal control problem for the adjoint system. We first state this for the finite horizon case, then we extend the result to the infinite horizon case.

Dual control problem: finite horizon: Define an *adjoint system*

$$\frac{d(F^T q(t))}{dt} = A^T q(t) - H^T u(t) \text{ and } F^T q(0) = F^T \ell \,. \tag{4}$$

Notation 2: $\forall \tau \in [0, +\infty]$ we set $I := [0, \tau] \cap [0, +\infty)$ and denote by $\mathscr{D}_\ell(I)$ the set of all $(q, u) \in L^2_{loc}(I, \mathbb{R}^m) \times L^2_{loc}(I, \mathbb{R}^p)$ such that $F^T q$ is absolutely continuous and (q, u) satisfy (4) almost everywhere.

Definition 4: We say that ℓ is a *consistent initial state* of (4), if there exists an interval $I = [0, \tau]$, or $I = [0, +\infty)$, such that (4) has a solution, i.e. $\mathscr{D}_\ell(I) \neq \emptyset$.
In [17, Lemma 2] it is shown that if ℓ is consistent, then $\mathscr{D}_\ell(I) \neq \emptyset$ for any interval I. Define $P = (\mathcal{I}_m - (F^T)^+ F^T)$, $\mathcal{M}_{opt} = P(PQ_0^{-1}P)^+ PQ_0^{-1} F^{T+}$ and $\bar{Q}_0 := (F^{T+} - \mathcal{M}_{opt})^T Q_0^{-1} (F^{T+} - \mathcal{M}_{opt})$.
For every $(q, u) \in \mathscr{D}_\ell(I)$, such that $[0, t_1] \subset I$, define

$$J(q, u, t_1) := \rho(F^T q(t_1), q, u, \tau, \bar{Q}_0, Q^{-1}, R^{-1}) \,.$$

The problem of finding $(q^*, u^*) \in \mathscr{D}_\ell([0, t_1])$ such that $J(q^*, u^*, t_1) = \inf_{(q,u) \in \mathscr{D}_\ell([0,t_1])} J(q, u, t_1) < +\infty$ is called the *finite horizon dual control problem* and (q^*, u^*) is called the solution of the finite horizon dual control problem.

Notation 3 (Shift δ_{t_1}): For any $v \in L^2(0, t_1)$, define $\delta_{t_1}(v) \in L^2(0, t_1)$ by $\delta_{t_1}(v)(s) = v(t_1 - s)$.

Theorem 1 (finite horizon duality principle): **1.** There exists a minimax observer if and only if ℓ is a consistent initial state of (4). **2.** If ℓ is consistent, then for any $U \in L^2(0, t_1)$,

$$\sigma(U, \ell, t_1) = \inf_{(q,v) \in \mathscr{D}_\ell([0,t_1]), v = \delta_{t_1}(U)} J(q, v, t_1) \,. \tag{5}$$

3. If ℓ is consistent, and (q^*, u^*) is a solution to the finite horizon dual control problem, then $\widehat{U} := \delta_{t_1}(u^*)$ is the minimax observer and $\sigma(\widehat{U}, \ell, t_1) = J(q^*, u^*, t_1) = \inf_{(q,u) \in \mathscr{D}_\ell([0,t_1])} J(q, u, t_1)$.

Proof: We extend the argument of [11, Theorem 2.4], where the generalized Kalman duality principle was proved for DAEs in the form (1) provided $x_0 = 0$, to the case of uncertain x_0. Let x solve (1) for $(x_0, f, \eta) \in \mathscr{E}(t_1)$. By using the integration by parts (see [11, F.(2.1)]) we find:

$$\ell^T F x(t_1) - \mathcal{O}_U(t_1, y) = x_0^T (F^+)^T F^T w(0) + \int_0^{t_1} (f^T w - \eta^T U + x^T b(w, U)) dt \tag{6}$$

for some $w \in L^2(0,t_1)$ such that $F^T w$ is absolutely continuous and $F^T w(t_1) = F^T \ell$, and $b(w,U)(t) := \frac{dF^T w(t)}{dt} + A^T w(t) - H^T U(t) \in L^2(0,t_1)$. Since $(x_0, f, \eta) \in \mathscr{E}(t_1)$, it follows from (6) that $\sigma(U,\ell,t_1) < +\infty$ if and only if

$$c(b,U) := \sup_{\{(x,\eta):(Lx,\eta)\in\mathscr{E}(t_1)\}} \int_0^{t_1} (x^T(t) b(w,U)(t) - \eta^T(t) U(t)) dt < +\infty$$

where L is the following differential operator associated to (1):

$$(Lx)(t) = \left(Fx(t_0), \frac{d(Fx)}{dt} - Ax(t) \right), x \in \mathscr{D}(L) := \{x \in L^2(0,t_1) \mid \frac{dFx}{dt} \in L^2(0,t_1)\}.$$

We note that $c(b,U)$ is the value of the support function of the pre-image of the ellipsoid $\mathscr{E}(t_1)$ with respect to the operator $(x,\eta) \mapsto (Lx,\eta)$ and $(\rho(\cdot,\cdot,\cdot,t_1,Q_0^{-1},Q^{-1},R^{-1}))^{\frac{1}{2}}$ is the support function of the ellipsoid $\mathscr{E}(t_1)$. Hence, from Young-Fenhel duality (see [11, F.(2.5),Lemma 2.2]),

$$c^2(b,U) = \inf_{g_0,g,v} \{\rho(g_0,g,v,t_1,Q_0^{-1},Q^{-1},R^{-1}) \mid (g_0,g) \in \mathscr{D}(L'), v \in L^2(0,t_1), \left[\begin{smallmatrix} L'(g_0,g) \\ v \end{smallmatrix}\right] = \left[\begin{smallmatrix} b(w,U) \\ U \end{smallmatrix}\right]\},$$

where L' is the adjoint of L and it is defined as follows:

$$L'(g_0,g)(t) = -\frac{dF^T g}{dt} - A^T g, (g_0,g) \in \mathscr{D}(L'),$$

$$\mathscr{D}(L') := \{g \in L^2(0,t_1) \mid \frac{dF^T g}{dt} \in L^2(0,t_1), F^T g(t_1) = 0, g_0 = F^{+T} F^T g(0) + d, F'd = 0\}.$$

Thus, $\sigma(U,\ell,t_1) < +\infty \Leftrightarrow c(b,U) < +\infty \Leftrightarrow L'(g_0,g)(t) = b(w,U)(t)$ for some $(g_0,g) \in \mathscr{D}(L')$ and so $z := w + g$ satisfies the following DAE: $\frac{d(F^T z(t))}{dt} = -A^T z(t) + H^T U(t)$, $F^T z(t_1) = F^T \ell$. Noting that the latter is equivalent to $(\delta_{t_1}(z), \delta_{t_1}(U)) \in \mathscr{D}_\ell([0,t_1])$ we obtain **1.**

Let us now prove **2.** From the discussion above, it follows that $\sigma(U,\ell,t_1) < +\infty$ if and only if there exists $(q,v) \in \mathscr{D}_\ell([0,t_1])$ such that $v = \delta_{t_1}(U)$. This means that if $\sigma(U,\ell,t_1) = +\infty$, then the set $\{(q,v) \in \mathscr{D}_\ell([0,t_1]), v = \delta_{t_1}(U)\}$ is empty and hence $\inf_{(q,v)\in\mathscr{D}_\ell([0,t_1]),v=\delta_{t_1}(U)} J(q,v,t_1) = +\infty$ and hence (5) holds. If $\sigma(U,\ell,t_1) < +\infty$, then choose $(p,u) \in \mathscr{D}_\ell([0,t_1]) \neq \varnothing$ with $U = \delta_{t_1}(u)$. Set $w := \delta_{t_1}(p)$. Then $b(w,U) = 0$ for $b(w,U)$ defined as above, and from (6):

$$\sigma^{\frac{1}{2}}(U,\ell,t_1) = c((F^+)^T F^T w(0), w, U) := \sup_{\mathscr{E}_1(t_1)} x_0^T (F^+)^T F^T w(0) + \int_0^{t_1} (f^T w - \eta^T U) dt, \quad (7)$$

where $\mathscr{E}_1(t_1)$ is composed of all $(x_0,f,\eta) \in \mathscr{E}(t_1)$ such that $Lx = (x_0,f)$ for some $x \in \mathscr{D}(L)$. Thus, to compute $\sigma(U,\ell,t_1)$ it is sufficient to find the support function c of $\mathscr{E}_1(t_1)$ which is defined by (7). The latter may be computed as follows (see [11, F.(2.6),Lemma 2.2]):

$$c^2((F^+)^T F^T w(0), w, U) = \inf_{(g_0,g)\in\ker L'} \rho((F^+)^T F^T w(0) + g_0, w + g, U, t_1, Q_0^{-1}, Q^{-1}, R^{-1}), \quad (8)$$

where $\ker L' = \{(g_0, g) \in \mathscr{D}(L') : L'(g_0, g) = 0\}$. Notice that $g_0 = F^{+T} F^T g(0) + P\tilde{d}$, $\tilde{d} \in \mathbb{R}^m$. Thus, for a fixed g, we can minimize ρ w.r.t. $\tilde{d}$ to find optimal $\hat{d}$. In fact, only the 1st term in ρ depends on $\tilde{d}$ and so, setting $\hat{d} := (PQ_0^{-1}P)^+ PQ_0^{-1}(F^+)^T F^T(w(0) + g(0))$, we get:

$$\inf_{\{g_0 | (g_0, g) \in \ker L'\}} \|Q_0^{-\frac{1}{2}}((F^+)^T F^T w(0) + g_0)\|^2 = \inf_{\tilde{d} \in \mathbb{R}^m} \|Q_0^{-\frac{1}{2}}((F^+)^T F^T(w(0) + g(0)) - P\tilde{d})\|^2$$
$$= \|Q_0^{-\frac{1}{2}}((F^+)^T F^T(w(0) + g(0)) - P\hat{d})\|^2 = (w(0) + g(0))^T F\bar{Q}_0 F^T(w(0) + g(0)) \,.$$

Now, let us note that $\{(q, v) \mid v = u, (q, v) \in \mathscr{D}_\ell([0, t_1])\} = \{(\delta_{t_1}(g + w), u) \mid (g_0, g) \in \mathscr{N}(L')\}$. In other words, any q solving (4) is a sum of $p = \delta_{t_1}(w)$ and δ_{t_1}-shift of a function g such that $L'(g_0, g) = 0$. This allows us to write for $\hat{g}_0 = F^{+T} F^T g(0) + P\hat{d}$:

$$\begin{aligned} &\inf_{\{g | (\hat{g}_0, g) \in \ker L'\}} (w(0) + g(0))^T F\bar{Q}_0 F^T(w(0) + g(0)) + \int_0^{t_1} (w + g)^T Q^{-1}(w + g) dt \\ &\qquad = \inf_{(q,u) \in \mathscr{D}_\ell([0,t_1]), u = \delta_{t_1}(U)} (F^T q(t_1))^T \bar{Q}_0 F^T q(t_1) + \int_0^{t_1} q^T Q^{-1} q dt \,. \end{aligned} \tag{9}$$

Now, (5) follows from (9), (8) and (7). Finally, to prove **3.** it is sufficient to apply Definition 1, that is to minimize (5) with respect to U. ■

Dual control problem: infinite horizon: For every $(q, u) \in \mathscr{D}_\ell([0, +\infty))$, define $J(q, u) := \limsup_{\tau \to \infty} J(q, u, \tau)$. The *infinite horizon dual control problem* is to find such $(q^*, u^*) \in \mathscr{D}_\ell([0, \infty))$ that

$$J(q^*, u^*) = \limsup_{\tau \to \infty} \inf_{(q,u) \in \mathscr{D}_\ell([0,\tau])} J(q, u, \tau) < +\infty. \tag{10}$$

The tuple (q^*, u^*) is called the solution of the infinite horizon dual control problem.

Theorem 2 (infinite horizon duality principle): Assume that ℓ is consistent and $(q^*, u^*) \in \mathscr{D}_\ell([0, +\infty))$ satisfies (10). Then $\widehat{U}(t_1, s) = u^*(t_1 - s)$ is the infinite horizon minimax observer.

Proof: Since ℓ is consistent, it follows by Theorem 1 that $\sigma(U, \ell, \tau) = \inf_{(q,v) \in \mathscr{D}_\ell([0,\tau]), v = u} J(q, v, \tau)$ for any $U \in L^2(0, t_1)$, $U = \delta_\tau(u)$. Hence

$$\sigma(U, \ell, \tau) = \inf_{(q,v) \in \mathcal{D}_\ell([0,\tau]), \delta_\tau(v) = U} J(q, v, \tau) \geq \inf_{(q,v) \in \mathcal{D}_\ell([0,\tau])} J(q, v, \tau) \,. \tag{11}$$

Assume that $\widehat{U}$ is defined as in theorem's statement. Let us prove that $\widehat{U}$ verifies (3). Take any $U \in \mathcal{F}$. Then by using (11) and the definition of (q^*, u^*), we get:

$$\sigma(U, \ell) = \limsup_{\tau \to \infty} \sigma(U(\tau, \cdot), \tau, \ell) \geq \limsup_{\tau \to \infty} \inf_{(q,v) \in \mathcal{D}_\ell([0,\tau])} J(q, v, \tau) = J(q^*, u^*) \,. \tag{12}$$

Since (11) holds for $\widehat{U}(\tau, \cdot)$, it follows that $\sigma(\widehat{U}(\tau, \cdot), \ell, \tau) = \inf_{(q,v) \in \mathcal{D}_\ell([\tau,0]), v = u_* |_{[0,\tau]}} J(q, v, \tau) \leq J(q^*, u^*, \tau)$, so that $\sigma(\widehat{U}, \ell) \leq \limsup_{\tau \to \infty} J(q^*, u^*, \tau) = J(q^*, u^*)$. Now, $\sigma(U, \ell) \geq J(q^*, u^*) \geq \sigma(\widehat{U}, \ell)$ by (12) and, $+\infty > J(q^*, u^*) \geq \sigma(\widehat{U}, \ell)$, so $\widehat{U}$ satisfies (3). ■

IV. Solution to the dual control problem

A. DAE systems as solutions to the output zeroing problem

We recall from [17] the definition of LTI systems whose outputs correspond to solutions (q, u) of (4). Let $r = \mathrm{Rank} F$. There exist nonsingular matrices S and T such that $SF^T T = \begin{bmatrix} \mathcal{I}_r & 0 \\ 0 & 0 \end{bmatrix}$. Consider the following decomposition

$$SA^T T = \begin{bmatrix} \widetilde{A} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \ SH^T = \begin{bmatrix} H_1^T \\ H_2^T \end{bmatrix}, \quad G = \begin{bmatrix} A_{12}, & -H_1^T \end{bmatrix}, \quad \widetilde{C} = A_{21} \text{ and } \widetilde{D} = \begin{bmatrix} A_{22}, & -H_2^T \end{bmatrix}.$$

$\widetilde{A} \in \mathbb{R}^{r \times r}$, $H_1 \in \mathbb{R}^{p \times r}$. Consider the following linear system

$$\mathcal{S} \begin{cases} \dot{p} = \widetilde{A} p + G v \\ z = \widetilde{C} p + \widetilde{D} v \end{cases}. \tag{13}$$

Now, it is not hard to see that if (q, u) solves (4) then there exists $v_1 \in L^2_{loc}(I, \mathbb{R}^{m-r})$ such that $T^{-1} q = (p^T, v_1^T)^T$, $v = (v_1^T, u^T)^T$, where (p, v) satisfy (13) with $z \equiv 0$. This intuition can be made more precise as follows. Recall from [18, Definition 7.8] the concept of the largest output nulling subspace $\mathcal{V} = \mathcal{V}(\mathcal{S})$, i.e. this is the largest subspace such that $p(0) \in \mathcal{V}$ if and only if there exists an input $v \in L^2_{loc}([0, +\infty), \mathbb{R}^{m-r+p})$, such that the corresponding output $z = z(p(0), v) \equiv 0$ almost everywhere. Recall from [18] that there exists a feedback map $\widetilde{F} \in \mathbb{R}^{m-r+p \times r}$ and an input transformation $L \in \mathbb{R}^{m-r+p \times k}$ for some $k > 0$ such that: $(\widetilde{A} + G\widetilde{F})\mathcal{V} \subseteq \mathcal{V}$ and $(\widetilde{C} + \widetilde{D}\widetilde{F})\mathcal{V} = \{0\}$, and $\mathrm{im} L = G^{-1}(\mathcal{V}) \cap \ker \widetilde{D}$, $\mathrm{Rank} L = k$. Define now the linear system $\mathscr{S} = (A_l, B_l, C_l, D_l)$ as follows. Choose a basis of $\mathcal{V} = \mathcal{V}(\mathcal{S})$ and let A_l, B_l, C_l be the matrix representations (in this basis) of the linear maps $(\widetilde{A} + G\widetilde{F}) : \mathcal{V} \to \mathcal{V}$, $GL : \mathbb{R}^k \to \mathcal{V}$, and $\bar{C} : \mathcal{V} \to \mathbb{R}^{m+p}$, and $D_l = \bar{D}$, where

$$\bar{C} = \begin{bmatrix} T & 0 \\ 0 & \mathcal{I}_p \end{bmatrix} \begin{bmatrix} \mathcal{I}_r \\ \widetilde{F} \end{bmatrix} : \mathcal{V} \to \mathbb{R}^{m+p} \ \text{ and } \bar{D} = \begin{bmatrix} T & 0 \\ 0 & \mathcal{I}_p \end{bmatrix} \begin{bmatrix} 0 \\ L \end{bmatrix} : \mathbb{R}^k \to \mathbb{R}^{m+p}.$$

Define C_s as the matrix formed by the first m rows of C_l, and let $\mathcal{M} = (F^T C_s)^+$.

Definition 5: The LTI $\mathscr{S} = (A_l, B_l, C_l, D_l)$ is called the LTI associated with the DAE (4), and $\mathcal{M}$ is called the corresponding map.

Theorem 3 (Theorem 1, [17]): Define $\mathcal{X} = \mathrm{im} F^T C_s$. For any interval $I = [0, +\infty)$ or $I = [0, t_1]$, the set $\mathcal{D}_\ell(I)$ is non-empty if and only if $F^T \ell \in \mathcal{X}$. If $(q, u) \in \mathcal{D}_\ell(I)$, then there exists an input $g \in L^2_{loc}(I, \mathbb{R}^k)$, such that $(q^T, u^T)^T = C_l v + D_l g$ a.e., provided $\dot{v} = A_l v + B_l g$ and $v(0) = \mathcal{M}(F^T \ell)$. Conversely, for any input $g \in L^2_{loc}(I, \mathbb{R}^k)$ there exists a solution $(q, u) \in \mathcal{D}_\ell(I)$ such that $(q^T, u^T)^T = C_l v + D_l g$, $\dot{v} = A_l v + B_l g$ and $v(0) = \mathcal{M}(F^T \ell)$.

Remark 1: Note that the LTI associated with (4) is not unique, due to the various choices of S, T, $\widetilde{F}$ and L and the basis of $\mathcal{V}$. However, all possible choices lead to linear systems which are feedback equivalent [17]. Furthermore, the largest output nulling subspace $\mathcal{V} = \mathcal{V}(\mathcal{S})$ can be computed by the Matlab toolbox of [19] or algorithm of [10]. Hence, the map $\mathcal{M}$ and the associated LTI $\mathscr{S}$ can be computed from (F, A, H).

Example 1: Consider the DAE (1), where

$$F = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix}, H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{14}$$

and take $Q_0 = Q = \mathcal{I}_2$, $R = \mathcal{I}_3$. In this case, the dual system is of the form

$$\frac{dz_1}{dt} = -z_2 - u_1, \quad z_1 \equiv 0, \quad \frac{dz_2}{dt} = -u_3, \quad -z_2 - u_2 = 0, z_1(0) = \ell_1, \quad z_2(0) = \ell_2\,. \tag{15}$$

It is then easy to see that with the notation of §IV-A, $\widetilde{A} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$, $G = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, $\widetilde{C} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\widetilde{D} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$. Then the largest output nulling subspace of the corresponding linear system is $\mathcal{V} = \{(0, z_2) \mid z_2 \in \mathbb{R}\}$ and the corresponding maps $\widetilde{F}$ and L are of the form $\widetilde{F} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \end{bmatrix}^T$ and $L = (0, 0, 1)^T$. Then the associated LTI $\mathscr{S} = (A_l, B_l, C_l, D_l)$ can be chosen as $A_l = 0$, $B_l = -1$, $C_l = (0, 1, -1, -1, 0)^T$ and $D_l = (0, 0, 0, 0, -1)^T$ and $\mathcal{M} = (0, 1, 0, 0, 0)$.

B. Dual control problem as a classical LQ

We apply Theorem 3 in order to solve the dual control problem. To this end, consider an associated LTI $\mathscr{S} = (A_l, B_l, C_l, D_l)$. For every initial state v_0, for every $t_1 > 0$, and every $g \in L^2_{loc}(I, \mathbb{R}^k)$, $[0, t_1] \subseteq I$, define the cost functional $\mathscr{J}(v_0, g, t_1)$:

$$\mathscr{J}(v_0, g, t_1) = \rho(F^T C_s v(t_1), q, u, t_1, \bar{Q}_0, Q^{-1}, R^{-1}) \tag{16}$$

$$\dot{v} = A_l v + B_l g, \quad v(0) = v_0, \quad (q^T, u^T)^T = C_l v + D_l g. \tag{17}$$

For any $g \in L^2_{loc}([0, +\infty), \mathbb{R}^k)$ and $v_0 \in \mathbb{R}^r$, define

$$\mathscr{J}(v_0, g) = \limsup_{\tau \to \infty} \mathscr{J}(v_0, g, \tau).$$

It is then easy to see that for any $g \in L^2_{loc}(I, \mathbb{R}^k)$, $[0, \tau] \subseteq I$ and for any ℓ such that $\mathcal{D}_\ell(I) \neq \emptyset$,

$$\mathscr{J}(\mathcal{M}(F^T \ell), g, \tau) = J(q, u, \tau) \text{ and } \mathscr{J}(\mathcal{M}(F^T \ell), g) = J(q, u) \tag{18}$$

where $(q^T, u^T)^T$ is defined by (17) with $v_0 = \mathcal{M}(F^T\ell)$. It then follows by Theorem 3 that $(q, u) \in \mathcal{D}_\ell(I)$, and, moreover, all $(q, u) \in \mathcal{D}_\ell(I)$ arise in this fashion for a suitable g.

Example 2: Recall Example 1. Noting that $\bar{Q}_0 = \mathcal{I}_2$ we obtain the following representation for the cost (16): $\mathscr{J}(v_0, g, t_1) = v^2(t_1) + \int_0^{t_1}(3v^2(t) + g^2(t))dt$.

C. Solution of the finite horizon optimal control problem

From (18) above it follows that in order to solve the finite horizon dual control problem , we need to solve the finite horizon control problem of finding g^* such that

$$\mathscr{J}(\mathcal{M}(F^T\ell), g^*, t_1) = \inf_{g \in L^2(I)} \mathscr{J}(\mathcal{M}(F^T\ell), g, t_1). \tag{19}$$

If g^* is the solution of (19), then the optimal input u^* is the output of the associated LTI from the initial state $\mathcal{M}(F^T\ell)$ under the input g^*. Since finding g^* amounts to solving the classical finite horizon LQ control problem, the next theorem follows from the classical results [18], [20].

Theorem 4 (Theorem 1, [17]): Assume that ℓ is a consistent initial state of (4). The finite horizon dual control problem (19) has the unique solution of the form:

$$\begin{aligned} &(q^{*T}(t), u^{*T}(t))^T = (C_l - D_l K(t_1 - t))v(t) \\ &\dot{v}(t) = (A_l - B_l K(t_1 - t))v(t) \text{ and } \quad v(0) = \mathcal{M}(F^T\ell). \end{aligned} \tag{20}$$

where $P(t)$ and $K(t)$ satisfy the following Riccati differential equation

$$\begin{aligned} &\dot{P}(t) = A_l^T P(t) + P(t)A_l - K^T(t)(D_l^T S D_l)K(t) + C_l^T S C_l \,, P(0) = (F^T C_s)^T \bar{Q}_0 F^T C_s \\ &K(t) = (D_l^T S D_l)^{-1}(B_l^T P(t) + D_l^T S C_l) \text{ and } S = \operatorname{diag}(Q^{-1}, R^{-1})\,. \end{aligned} \tag{21}$$

D. Infinite horizon dual control problem

Analogously to the finite horizon case, we can reduce the infinite horizon dual control problem to the following infinite horizon LQ problem. Let $\mathscr{S} = (A_l, B_l, C_l, D_l)$ denote the LTI associated with (4) and let $\mathcal{M} = F^T C_s$ be the corresponding map. Following [18] we introduce the stabilizability subspace $\mathcal{V}_g$ of $\mathscr{S}$. That is, $\mathcal{V}_g$ is the largest set of all initial states for which there exists an input g such that $\lim_{t\to\infty} v(t) = 0$ for the corresponding state trajectory $v(t)$. From [21] it then follows that $\mathcal{V}_g$ is A_l-invariant and $\mathrm{im} B_l \subseteq \mathcal{V}_g$. Now, we take a basis $b_1, \dots, b_r$ in the state-space of $\mathscr{S}$ such that $b_1, \dots, b_l$, $l = \dim \mathcal{V}_g$, span $\mathcal{V}_g$. In this basis,

$$A_l = \begin{bmatrix} \hat{A}_g & \star \\ 0 & \star \end{bmatrix}, B_l = \begin{bmatrix} \hat{B}_g \\ 0 \end{bmatrix}, C_l = \begin{bmatrix} \hat{C}_g & \star \end{bmatrix}, \mathcal{M} = \begin{bmatrix} \mathcal{M}_g \\ \star \end{bmatrix}$$

Denote by $\mathscr{S}_g = (\hat{A}_g, \hat{B}_g, \hat{C}_g, \hat{D}_g)$, where $\hat{D}_g = D_l$. The LTI $\mathscr{S}_g$ represents the restriction of $\mathscr{S}$ to the subspace $\mathcal{V}_g$. From [17] it follows that $\mathscr{S}_g$ is stabilizable.

Definition 6 (Stabilizable associated LTI): We call the LTI $\mathscr{S}_g$ defined above the stabilizable LTI associated with (4) and we call $\mathcal{M}_g$ the associated map.

Using (18) and the classical results on infinite horizon LQ control, we obtain the following.

Theorem 5 (Theorem 5, [17]): The following are equivalent:

- **(i)** The infinite horizon dual control problem is solvable for ℓ.
- **(ii)** $\mathcal{M}(F^T\ell)$ belongs to the stabilizability subspace $\mathcal{V}_g$ of the associated LTI $\mathscr{S}$.
- **(iii)** $\limsup_{t_1\to\infty} \inf_{(q,u)\in\mathcal{D}_\ell([0,t_1])} J(q,u,t_1) < +\infty$.

If $\mathcal{M}(F^T\ell) \in \mathcal{V}_g$, then there exists a unique positive definite matrix P such that:

$$
\begin{aligned}
&0 = PA_g + A_g^T P - K^T(D_g^T S D_g)K + C_g^T S C_g \\
&K = (D_g^T S D_g)^{-1}(B_g^T P + D_g^T S C_g) \text{ and } S = \mathrm{diag}(Q^{-1}, R^{-1})
\end{aligned}
\tag{22}
$$

and $A_g - B_g K$ is a stable matrix. Let v solve

$$\dot{v} = (A_g - B_g K)v\,, \quad v(0) = \mathcal{M}_g(F^T\ell)\,.$$

It then follows that $(q^{*T}, u^{*T})^T = (C_g - D_g K)v$ is a solution of the infinite horizon dual control problem, and $J(q^*, u^*) = \ell^T F \mathcal{M}_g P \mathcal{M}_g F^T \ell\,.$

E. Observer design for DAE

Now we present the main result on existence of the minimax observer and the corresponding algorithm. To this end, we introduce the following detectability and observability notions.

Definition 7 (Detectability): We say that the tuple (F, A, H) is ℓ-*observable*, if ℓ is a consistent initial state of the adjoint system (4). The tuple (F, A, H) is said to be ℓ-*detectable*, if the dual system (4) has a solution $(q, u) \in \mathcal{D}_\ell([0, +\infty))$ such that $\lim_{t\to\infty} F^T q(t) = 0$. The tuple (F, A, H) is *observable* (resp. *detectable*), if it is ℓ-observable (resp. ℓ-detectable) for all $\ell \in \mathbb{R}^m$.

Clearly, ℓ-detectability implies ℓ-observability. The reverse might not hold true, though. The notion of ℓ-detectability is closely related to the stabilizability of the LTI system associated with (4). Notice that according to the terminology of [17], ℓ-observability is the same as ℓ being differentiably consistent initial state of the adjoint system (4), and ℓ-detectability is the same as stabilizability from ℓ. Having this in mind we reformulate Lemma 3 from [17] as follows:

Lemma 1 (Lemma 3, [17]): Consider an LTI $\mathscr{S}$ associated with the dual system (4) and consider the corresponding map $\mathcal{M}$. Then (F, A, H) is ℓ-detectable if and only if $\mathcal{M}(F^T\ell)$ belongs to the stabilizability subspace $\mathcal{V}_g$ of $\mathscr{S}$.

The motivation for introducing the notions of ℓ-observability and ℓ detectability is that they are necessary and sufficient conditions for existence of the finite and infinite minimax observers respectively. For the finite horizon case, Theorem 4 and Theorem 1 imply the following.

Theorem 6 (Minimax observer: finite horizon): Fix $t_1 > 0$. The finite horizon minimax observer $\widehat{\mathcal{O}}_{\widehat{U}}(t_1, y)$ exists if and only if (F, A, H) is ℓ-observable. If (F, A, H) is ℓ-observable, then the minimax observer $\widehat{\mathcal{O}}_{\widehat{U}}(t_1, y)$ on $[0, t_1]$ can be computed as follows.

- **Step 1.** Compute the LTI associated with the dual system (4).
- **Step 2.** Solve the Riccati differential equation (21).
- **Step 3.** Set $\widehat{U}(t) = u^*(t_1 - t)$, where u^* is as in (20). Then $\widehat{\mathcal{O}}_{\widehat{U}}(t_1, y) = \ell^T F \mathcal{M}^T r(t_1)$ and the minimax error is $\sigma(\widehat{U}, \ell, t_1) = \ell^T F \mathcal{M}^T P(t_1) \mathcal{M} F^T \ell$ where r solves

$$\dot{r}(t) = (A_l - B_l K(t))^T r(t) + (C_l - D_l K(t))^T \begin{bmatrix} 0 \\ y(t) \end{bmatrix} \text{ and } r(0) = 0\,. \tag{23}$$

Proof: Let v be the solution of (20) and let r be the solution of (23). An easy calculation reveals that $\frac{d}{dt}(r^T(t)v(t_1 - t)) = y^T(t)u^*(t_1 - t)$, from which it follows that $\widehat{\mathcal{O}}_{\widehat{U}}(t_1, y) = \int_0^{t_1} y^T(t)u^*(t_1 - t) = r^T(t_1)v(0) - r^T(0)v(t_1) = \ell^T F \mathcal{M}^T r(t_1)$. The rest of the theorem follows easily from Theorem 4 and Theorem 1. ∎

A similar theorem can be stated for the infinite horizon case.

Theorem 7 (Minimax observer: infinite horizon): The infinite horizon minimax observer $\widehat{\mathscr{O}}_{\widehat{U}}(y)$ for the DAE (1) exists, if and only if the DAE (1) is ℓ-detectable. If the DAE (1) is ℓ-detectable, then the minimax observer $\widehat{\mathscr{O}}_{\widehat{U}}(y)$ can be computed as follows.

- **Step 1.** Construct the stabilizable LTI $\mathscr{S}_g = (A_g, B_g, C_g, D_g)$ associated with the dual system (4) and the map $\mathcal{M}_g$.
- **Step 2.** Find the unique matrices $P > 0$ and K which satisfy the Riccati equation (22).
- **Step 3.** The infinite horizon minimax observer is of the form: $\widehat{\mathscr{O}}_{\widehat{U}}(y)(t) = \ell^T F \mathcal{M}_g^T r(t)$ and the minimax error is $\sigma(\widehat{U}, \ell) = \ell^T F \mathcal{M}_g^T P \mathcal{M}_g F^T \ell$, where

$$\dot{r}(t) = (A_g - B_g K)^T r(t) + (C_g - D_g K)^T \begin{bmatrix} 0 \\ y(t) \end{bmatrix} \text{ and } r(0) = 0\,. \tag{24}$$

Proof: If (F, A, H) is ℓ-detectable, then $\mathcal{M}(F^T\ell)$ belongs to the stabilizability subspace of the associated LTI $\mathscr{S}$ and hence by Theorem 2 and Theorem 5, the tuple (q^*, u^*) described in Theorem 5 is a solution of the infinite horizon dual control problem. Therefore, $\widehat{U}(t,s) = u^*(t-s)$ is the minimax observer by Theorem 2. From Theorem 5 it follows that

$$u^*(t) = \begin{bmatrix} 0 & 0 \\ 0 & I_p \end{bmatrix} (C_g - D_g K) e^{(A_g - B_g K)t} \mathcal{M}_g F^T \ell \,.$$

To prove that $\widehat{\mathscr{O}}_{\widehat{U}}(y)(t) = \ell^T F \mathcal{M}_g^T r(t)$ we note that:

$$\widehat{\mathscr{O}}_{\widehat{U}}(y)(\tau) = \int_0^\tau u_*^T(\tau - t) y(t) dt = \ell^T F \mathcal{M}_g^T \int_0^\tau e^{(A_g - B_g K)^T(\tau - t)} (C_g - D_g K)^T \begin{bmatrix} 0 \\ y(t) \end{bmatrix} dt \,.$$

Finally, we show that existence of a minimax observer implies ℓ-detectability. Let $\widehat{U}$ be an infinite horizon minimax observer. Then from Theorem 1, $\inf_{(q,u)\in\mathcal{D}_\ell([0,t_1])} J(q,u,t_1) = \inf_{v\in L^2(0,t_1)} \sigma(v,\ell,t_1) \le \sigma(\widehat{U}(t_1,\cdot),\ell,t_1)$, and hence

$$\begin{aligned} \limsup_{t_1\to\infty} \inf_{(q,u)\in\mathcal{D}_\ell([0,t_1])} J(q,u,t_1) &= \limsup_{t_1\to\infty} \inf_{v\in L^2(0,t_2)} \sigma(v,\ell,t_1) \\ &\le \limsup_{t_1\to\infty} \sigma(\widehat{U}(t_1,\cdot),\ell,t_1) = \sigma(\widehat{U},\ell) < +\infty. \end{aligned} \tag{25}$$

Consider the associated LTI $\mathscr{S} = (A_l, B_l, C_l, D_l)$ and the map $\mathcal{M}$. From Theorem 5, **(iii)** and (25) it then follows that $\mathcal{M}(F^T\ell)$ belongs to the stabilizability subspace $\mathcal{V}_g$ of $\mathscr{S}$ and hence by Lemma 1, (F, A, H) is ℓ-detectable. ■

Remark 2: Theorem 7 indicates that ℓ-detectability is necessary and sufficient for the existence of the minimax observer. In particular, observability implies the existence of the finite horizon minimax observer and detectability implies the existence of the infinite horizon minimax observer for all $\ell \in \mathbb{R}^m$.

Remark 3: For the case $f = 0$ and $\eta = 0$ it is not hard to prove by using the duality argument that $\lim_{t\to\infty}(\ell^T F x(t) - \widehat{\mathscr{O}}_{\widehat{U}}(y)(t)) = 0$ where $y \in L^2_{loc}([0,+\infty),\mathbb{R}^p)$ is the output of (1).

Example 3: Recall Example 1. It then follows that $\ell = (\ell_1, \ell_2)^T$ is a consistent initial state of the adjoint system if and only if $\ell_1 = 0$. Specifically, $\ell = (0,1)^T$ is a consistent initial state and $\mathcal{M}(F^T\ell) = 1$. In this case, the associated LTI $\mathscr{S} = (A_l, B_l, C_l, D_l)$ is controllable and hence its stabilizability subspace equals its state-space. In particular, the DAE (1) is ℓ-detectable, $\mathscr{S} = \mathscr{S}_g$ and $\mathcal{M} = \mathcal{M}_g$. (21) becomes $\dot{P}(t) = -P(t)^2 + 3$, $K(t) = P(t)$, $P(0) = 1$ and (22) becomes $-P^2 + 3 = 0$, $K = P$, from which it follows that $K = \sqrt{3}$. It then follows that

$A_l - B_l K = -\sqrt{3}$, $C_l - D_l K = (0, 1, -1, -1, \sqrt{3})$ and $\ell^T F \mathcal{M}^T = 1$. Hence, $\widehat{\mathscr{O}}_{\widehat{U}}(y)(t) = r(t)$ where $\dot{r}(t) = -\sqrt{3} r(t) - y_1(t) - y_2(t) + \sqrt{3} y_3(t)$. The minimax error is given by $\sqrt{3}$.

Remark 4 (Relationship with [15], [22]): In [15] it was shown that impulse observability and solvability of a certain LMI is a sufficient condition for existence of a robust observer, which minimizes the H_∞ norm of the estimation error for all ℓ. From [23] it follows that the condition of impulse observability is equivalent to impulse controllability of the adjoint DAE (4), and the latter is equivalent to ℓ-observability for all ℓ. Note that the minimax error $\sigma(U, \ell)$ of the minimax observer U is different from the H_∞ norm of the error system of [15], [22]. Indeed, the H_∞ norms gives the L^2-norm of the estimation error subject to noise of unit energy, while in our case we look at the limit supremum of the observation error.

In [22] an asymptotic functional observer was proposed for the noiseless case, i.e. for the case when $f = 0, \nu = 0$. There it was shown that partial impulse observability and certain additional conditions are sufficient for existence of an observer. In contrast to [22], we deal with noisy systems, moreover, our conditions are necessary and sufficient. It remains a future work to relate ℓ-detectability with the conditions of [22].

V. Conclusions

We have presented a solution of the minimax observer design problem for linear DAEs. The proposed framework is "application friendly" as it imposes just necessary and sufficient conditions (ℓ-detectability) on the matrices (F, A, H), handles L^2-noise and provides an algorithm for constructing the optimal minimax observer for any choice of the associated system. The case of L^∞-noise is left for the future work.

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