Robust source estimation for wave equations

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Governing Equation

Given a computational domain $\Omega \subset \mathbb{R}^3$ with boundary Γ, the elastic wave equation is defined as:

$$\rho\partial_{tt}u_i = \partial_{x_j}\sigma_{ij} + f_i$$

where $x_i \in \Omega$ (summation is implied over the repeated index), $u(x_i,t)$ is the displacement vector field, $\rho(x_i)$ is the density, $f(x_i,t)$ is the seismic source, and $\sigma_{ij}(x_i)$ is the stress tensor, modeled with the linear constitutive relation $\sigma_{ij} = c_{ijkl}\epsilon_{kl} = \lambda\epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij}$, and $\lambda(x_i)$ and $\mu(x_i)$ are the first and second *Lamé* parameters respectively [1].

Problem statement

With the Spectral Element Method, we arrive at a system of ODEs:

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{s}\,, \tag{1}$$

where M, C, K are the mass, damping and stiffness matrices respectively, $\mathbf{u}$ is a discrete representation of the wavefield u, $\mathbf{s}$ stands for a source vector. We assume that a vector $Y(t)$ is observed in the following form:

$$Y(t) = H\mathbf{u} + e(t)\,, \tag{2}$$

where $e(t)$ denotes the observation error and the matrix H describes the locations of the receivers in Ω. We further assume that $e(t)$ is a realisation of a random process with zero mean and a given covariance function $R(t)$. Given the data Y, we would like to estimate the position and intensity of the source term s assuming that:

$$\boldsymbol{s} = \boldsymbol{B}\boldsymbol{f}$$

and $\dot{\mathbf{f}} = \mathbf{g}(t)$ where $\boldsymbol{B}$ is a matrix representing a possible spacial localisation of the source term (namely, each column of this matrix represents a particular spacial location in the domain Ω). Vector function $\mathbf{g}$ has bounded energy, and satisfies the following inequality:

$$\int_0^{t^*} \|\mathbf{g}(t)\|^2 dt \leqslant \mu^2\,,$$

where μ is a given scalar.
By extending (1) with an additional equation:

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{u}} &= \boldsymbol{B}\boldsymbol{f} + \mathbf{g} - \mathbf{C}\dot{\mathbf{u}} - \mathbf{K}\mathbf{u}\,, \quad \mathbf{u}(0) = \dot{\mathbf{u}} = 0\,,\\ \dot{\mathbf{f}} &= g(t)\,, \quad \mathbf{f}(0) = 0\,. \end{aligned} \tag{3}$$

we convert the source estimation problem into a state estimation problem: estimate $\mathbf{B}\boldsymbol{f}(t^*)$, taking into account observations, $Y(t)$, and a non-standard uncertainty description (random measurement's noise and deterministic model error).

Minimax state estimation

We construct a minimax filter [4] to estimate $\mathbf{f}(t^*)$. Namely, we define an extended state vector $x(t) = (\mathbf{u}, \dot{\mathbf{u}}, \mathbf{f}(t))^\top$ and matrices

$$A := \begin{bmatrix} 0 & I & 0 \\ -M^{-1}K & -M^{-1}C & M^{-1}B \\ 0 & 0 & 0 \end{bmatrix}, \quad H_e = \begin{bmatrix} H & 0 & 0 \end{bmatrix}.$$

Then (3) is equivalent to the following system of ODEs:

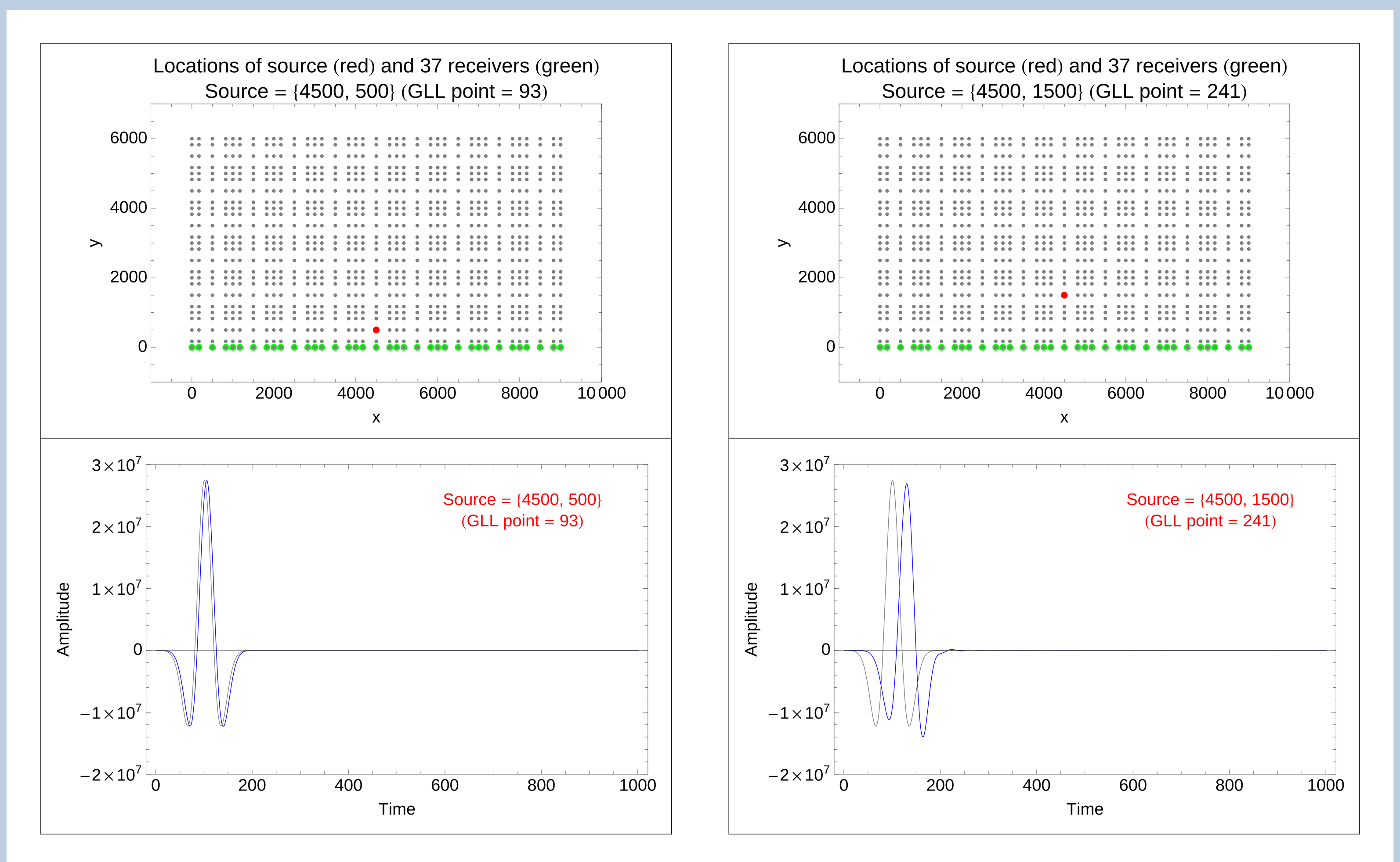
$$\dot{x} = Ax + Wg\,, W = \begin{bmatrix} 0 & I \end{bmatrix}\,,$$

and minimax estimate is given by:

$$\dot{\hat{x}} = A\hat{x} + UV^{-1}RH_e(Y(t) - H_e\hat{x})\,, \tag{4}$$

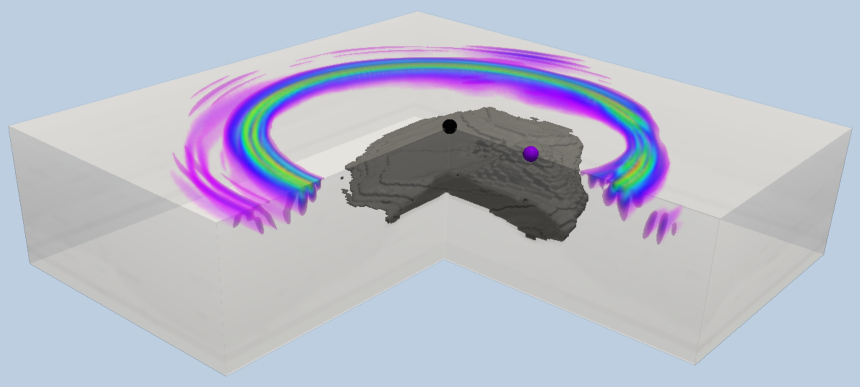
$$\begin{pmatrix} dU/dt \\ dV/dt \end{pmatrix} = \begin{bmatrix} -A^T & H_e^T R H_e \\ W\mu^{-1}W^\top & A \end{bmatrix} \begin{pmatrix} U \\ V \end{pmatrix}.$$

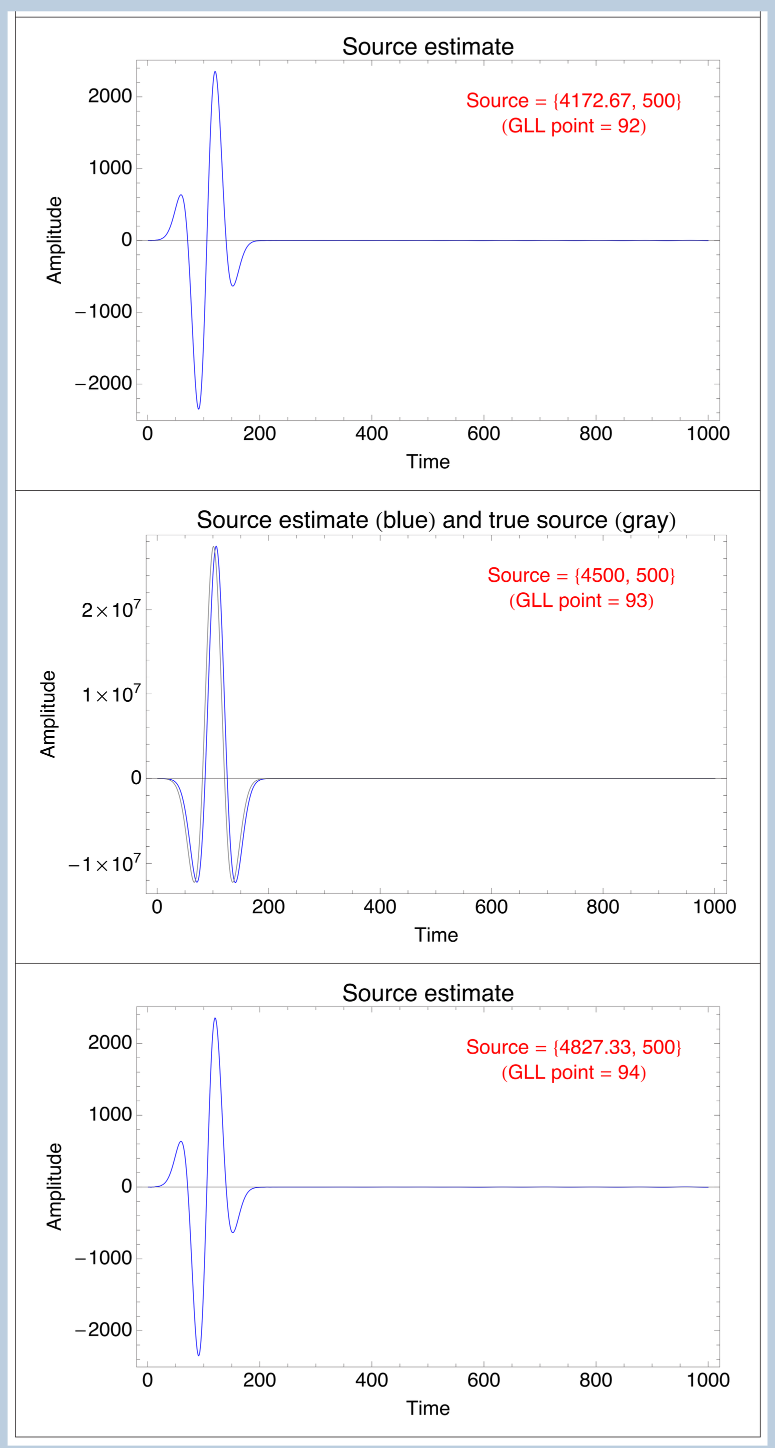
Results for 2D acoustic equation



To construct a numerical approximation for (4), we note that (3) is stiff and so implicit methods are preferable. We compute minimax gain $P = VU^{-1}$ and minimax estimate $\hat{x}$ by using a Möbius transform combined with implicit midpoint. The eigenvalues of $P(t)$ define the size of the ellipsoid containing $x(t)$ and so to get reliable error estimates, the chosen numerical method preserves quadratic invariants [3]. As a result we obtain a guaranteed estimate of the state vector $x(t) = (\mathbf{u}, \dot{\mathbf{u}}, \mathbf{s}(t))^\top$. The presented figure shows the estimation results for the case of 2D acoustic wave equation and given source position when subject to a standard Ricker wavelet source function. The source estimate (bottom left) is almost exact if the recievers are quite close to the actual source location (top left) and there is a time delay in the intensity tracking (bottom right) if the recievers are further away (top right) from the source.

Results for 3D elastic equation





Presented here are estimation results for the case of 3D elastic wave equation (top left) and assuming that the source position is unknown but is in either of 3 given locations. The reciever's are located close to the source. The algorithm properly identifies actual source location (middle left) out of 3 possible locations and tracks the intensity of the source. The other two locations receive low weights (compared to the one in the middle).
The simulations were performed on an IBM Blue Gene/Q using on a grid defined by approximately sixteen thousand 5^{th} order elements, resulting in a total of approximately 6.5 million degrees of freedom. The parallelization strategy is based upon decomposition of the grid across multiple processes such that each process contains a unique set of the global GLL points and hence contiguous rows of the global mass, damping, stiffness matrices. The explicit construction of a stiffness matrix implies that the time marching update can be performed by a distributed matrix vector multiplication at each time step and to to so the implementation makes use of the Watson Sparse Matrix Package (WSMP) [2] in order to provide a scalable code allowing for hybrid parallelization based on multithreading and the message passing interface (MPI).

References

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[3] J. Frank and S. Zhuk, *Symplectic möbius integrators for LQ optimal control problems*. CDC (2014).
[4] S. Zhuk, *Estimation of the states of a dynamical system described by linear equations with unknown parameters*, Ukrainian Math. J., 61 (2009)