**Governing Equation**

Given a computational domain $\Omega \subset \mathbb{R}^3$ with boundary $\Gamma$, the elastic wave equation is defined for $x_i \in \Omega$ as [1]:

$$\rho \partial_t u_i = \partial_j \sigma_{ij} + f_i$$

$u(x_i, t)$ displacement field [m]  
$\rho(x_i)$ density field $[\text{kg m}^{-3}]$  
$f(x_i, t)$ seismic source $[\text{Nm}^{-1}]$  
$\sigma_{ij}(x)$ stress tensor $[\text{Pa}]$  
$\lambda(x)$ Lamé first parameter $[\text{Pa}]$  
$\mu(x)$ Lamé second parameter $[\text{Pa}]$

**Spatial Discretization**

Using the Spectral Element Method, the computational domain is defined as a tesselation of hexahedral elements with the displacement field approximated as $u_i(x_i, t) \approx \psi_{i,pqr}(x_i)$, which combined with Galerkin projection gives the weak form:

$$\int_{\Omega} \psi_{i,pqr} \rho \partial_t u_{i,pqr} d\Omega + \int_{\Omega} \psi_{i,pq} \mu \partial_j \psi_{i,qr} \partial_j u_{i,qr} d\Omega +$$

$$\int_{\Omega} \partial_j \psi_{i,ab} c_{ijkl} \left( \frac{1}{2} (\partial_j \psi_{i,qr} \partial_k u_{j,qp} + \partial_k \psi_{i,qr} \partial_j u_{j,qp}) \right) d\Omega =$$

$$\int_{\Omega} \psi_{i,pq} f_{i,pq} d\Omega$$

Here, the basis functions $\psi_{i,pq}$ are taken to a tensor product of the family of $N^{th}$ order Lagrange polynomials $\psi_{i,pq} = \psi_{i,p} \psi_{q}$, which integrating over a reference element with Gauss-Legendre-Lobatto (GLL) quadrature [2], gives the elemental matrices:

$$M^{e}_{i,j,abcd,pq} = \sum_{p=1}^{N+1} \sum_{q=1}^{N+1} w_p w_q \psi_{i,ab} \psi_{j,c} \psi_{k,d} \psi_{l,pq} J_{1,ef,gh,kl}$$

$$C^{e}_{i,j,abcd,pq} = \sum_{p=1}^{N+1} \sum_{q=1}^{N+1} w_p w_q \psi_{i,ab} \psi_{j,c} \psi_{k,d} \psi_{l,pq} \frac{\partial \psi_{i,ab} \partial \psi_{j,c} \partial \psi_{k,d} \partial \psi_{l,pq}}{\partial x_p \partial x_q \partial x_r \partial x_s} J_{1,ef,gh,kl}$$

$$K^{e}_{i,j,abcd,pq} = \sum_{p=1}^{N+1} \sum_{q=1}^{N+1} w_p w_q \psi_{i,ab} \psi_{j,c} \psi_{k,d} \psi_{l,pq} \frac{\partial \psi_{i,ab} \partial \psi_{j,c} \partial \psi_{k,d} \partial \psi_{l,pq}}{\partial x_p \partial x_q \partial x_r \partial x_s} \frac{\partial \psi_{i,ab} \partial \psi_{j,c} \partial \psi_{k,d} \partial \psi_{l,pq}}{\partial x_p \partial x_q \partial x_r \partial x_s} J_{1,ef,gh,kl}$$

where $J$ is the Jacobian of the transformation between the global coordinates $x_i$ and the reference.

**Temporal Discretization**

The assembly of the elemental matrices into their global counterparts, defines the system of ordinary differential equations:

$$Mu + Cu + Ku = s$$

where $M$, $C$, $K$, and $s$ are the global mass, damping, stiffness matrices, and source vector. Using the Newmark method [3], the system is marched forward in time as:

$$\ddot{u}^{n+1} + \gamma \dot{u}^{n+1} + \ddot{u}^{n} = \ddot{s}_{n+1}$$

$$\ddot{u}^{n+1} = \ddot{u}^{n} + \Delta t \gamma (\dddot{u}^{n} + \dddot{u}^{n+1})$$

$$\ddot{u}^{n+1} = \dddot{u}^{n+1} + \Delta t (1 - \beta) \dddot{u}^{n+1} + \frac{\Delta t^2}{2} (2\dddot{u}^{n} + \dddot{u}^{n+1})$$

where the choice of $\gamma = 0.5$ and $\beta = 0$, results in an explicit time marching scheme.

**Parallelization**

The parallelization strategy is based upon decomposition of the grid across multiple processes such that each process contains a unique set of the global GLL points and hence contiguous rows of the global mass, damping, stiffness matrices. The explicit construction of a stiffness matrix implies that the time marching update can be performed by a distributed matrix vector multiplication at each time step and to do so the implementation makes use of the Watson Sparse Matrix Package (WSMP) [4] in order to provide a scalable code allowing for hybrid parallelization based on multithreading and the message passing interface (MPI).

**Results**

Presented below is a sample forward simulation on the SEG/EAGE salt model dataset, illustrating the magnitude of the displacement field and the receiver traces when subject to a standard Ricker wavelet source function. The simulation was performed on IBM Blue Gene/Q using a grid of approximately 65 thousand 5th order elements, resulting in a total of approximately 25 million degrees of freedom.

**Future Work**

This forward solver is the first addition towards the ongoing development of a modular framework of scalable full waveform inversion tools. The delivery model will be through a cloud based service where users will have the ability to upload seismographic datasets and corresponding geographic coordinates through a web based interface. Using these inputs in combination with the desired physical model, the appropriate modules will be selected for grid generation, spatial and temporal discretization of the selected forward model, inversion, and the large scale simulations will be performed on the appropriate High Performance Computing (HPC) resource [5].

**References**