Kalman duality principle for a class of ill-posed minimax control problems with linear differential-algebraic constraints

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Abstract. In this paper we present Kalman duality principle for a class of linear Differential-Algebraic Equations (DAE) with arbitrary index and time-varying coefficients. We apply it to an ill-posed minimax control problem with DAE constraint and derive a corresponding dual control problem. It turns out that the dual problem is ill-posed as well and so classical optimality conditions are not applicable in the general case. We construct a minimizing sequence $\hat{u}_\varepsilon$ for the dual problem applying Tikhonov method. Finally we represent $\hat{u}_\varepsilon$ in the feedback form using Riccati equation on a subspace which corresponds to the differential part of the DAE.

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1. Introduction

In this paper we treat the solution of the following optimization problem:

$$\mathscr{I}(u) := \int_{t_0}^{t_1} u^T R(t) u dt + \sup_{f \in \mathscr{G}} \{\mathcal{L}(x,u)\}^2 \to \inf_{u \in \mathbb{L}_2(t_0,t_1,\mathbb{R}^p)} \tag{1.1}$$

$$\frac{d(Fx)}{dt} = C(t)x(t) + f(t), \quad Fx(t_0) = 0\,, \tag{1.2}$$

where R is a symmetric positive definite continuous matrix, $\mathcal{L}(\cdot,\cdot)$ is a linear functional, $\mathscr{G} \subset \mathbb{L}_2(t_0,t_1,\mathbb{R}^m)$, F is a $m \times n$-matrix and C is a continuous matrix. The $\sup\{\cdot\}$ in (1.1) is taken over all $f(\cdot)$ and $x(\cdot)$ such that $f(\cdot) \in \mathscr{G}$ and $x(\cdot)$ is one of solutions of (1.2) corresponding to $f(\cdot)$.

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The problem (1.1)-(1.2) arises in state estimation theory where one aims to estimate (interpolate, filter or predict) the state vector $x(t_1)$ of the Differential-Algebraic Equation (DAE) (1.2) or its linear combinations $Fx(t_1)$ (see [1, 2, 3] for details), given measurements of the state vector $x(t)$ in the past. In this context, the cost (1.1) defines the quality of the estimate: for instance, in the minimax state estimation framework it defines a so called worst-case state estimation error. We stress that in the classical state estimation setting (see [4, 5, 6, 7] for details) the state equation is usually represented by an Ordinary Differential Equation (ODE) or Partial Differential Equation (PDE). Description of the state equation by means of DAE is motivated by a practical application of minimax methods for high-dimensional systems of ODEs resulting from the discretization of PDEs. Classical state estimation algorithms such as Kalman filter [8, 7], minimax filters [4, 5, 6, 9] or H_∞ filters [10] may be so demanding in terms of computations that they cannot be applied to high-dimensional ODEs without an appropriate reduction. One way to reduce a dimension is inspired by Galerkin method, that is to project the state vector of a high-dimensional (or infinite dimensional in the case of PDE) model onto a low-dimensional subspace[1] and describe the dynamics of the projection's coefficients. However, the projection introduces errors that can lead to a reduced equation, describing the time evolution of the projection's coefficients, with unstable dynamics [2]. In order to address this issue, an additional energy constraint on the projection's coefficients is introduced in the form of a linear algebraic equation. Roughly speaking, this constraint allows to keep the reduced model state's energy in the prescribed bounds. As a result, the reduced model is represented by a DAE (see [2] for details) and the suitable state estimation approach is applied to this DAE. Various applications of DAEs (1.2) may be found in robotics (constrained dynamics of multibody systems [12]), filtering (reduced order models [13]) and circuits theory [14]. Pros and cons of using DAEs for modeling were discussed in [15].

The set of all solutions of (1.2) was described in [16] by means of the theory of matrix pencils, provided $C(t) \equiv C$. The main idea behind this approach is to transform the pencil $F - \lambda C$ to Weierstrass canonical form. This allows to convert DAE (1.2) into equivalent system of ODEs with compatibility conditions for the data $f(\cdot)$ in the form of algebraic constraints. Conditions on $f(\cdot)$ reveal the ill-posedness of (1.2) in $\mathbb{L}_2$ (see [17]): the solution of (1.2) may not exist or may not be uniquely defined or may depend on derivatives of $f(\cdot)$. If $C(t)$ is not constant then the matrix pencil theory applies if the degree of the polynomial $\det(F-\lambda C(t))$ is constant for all $t \in [t_0, t_1]$ (see [18]). It may fail, however, in the general case (see [19]). Possible ways of converting DAE into ODE for the case $m = n$ and variable F, C were discussed in [20].

Classical Kalman duality principle states that (1.1)-(1.2) is equal to the dual control problem for adjoint equation, provided $F = I$ and f is a so called "white

[1] This subspace might be generated by means of a low-rank approximation technique like Proper Orthogonal Decomposition.

noise” (see [21]). In this paper we generalize it to the case $F \in \mathbb{R}^{m\times n}$ and $f(\cdot) \in \mathscr{G}$. A direct way to perform this generalization would be to convert (1.2) into ODE and apply the classical duality principle. As noted above, a transformation of the DAE into ODE has a major drawback: one might need to differentiate $f(\cdot)$. Although smoothness of $f(\cdot)$ may be appropriate[2] for control problems where $f(\cdot)$ stands for the control parameter, it turns out to be a very restrictive assumption in the context of state estimation where $f(\cdot)$ is the model error which is often represented by a random process in stochastic filtering or measurable function with bounded $\mathbb{L}_2$-norm in the minimax framework. In this situation one may differentiate $f(\cdot)$ in the weak sense but the solution of DAE (1.2) becomes then a distribution. However, the latter is not desirable in the case when the solution of DAE (1.2) describes dynamics of projection coefficients of an absolutely continuous function solving a high-dimensional ODE (as it was discussed above) and, thus, is at least of $\mathbb{L}_2$ class. This argument motivated us to apply operator theory in order to treat DAEs in the form (1.2) without trying to convert it into ODE. We refer the reader to [23, 24, 25] for details on operator methods and their application to differential equations. Our approach is as follows: we represent DAE (1.2) as an operator equation $Lx = f$ with a linear closed unbounded mapping L in a special Hilbert space. This interpretation allows us to deal with weak solutions of (1.2) belonging to a special Sobolev space and take into account high-index DAEs. Then, noting that $\sup\{\cdot\}$ in (1.1) may be thought of as a support function of the set $L^{-1}(\mathscr{G})$, we compute it applying Young-Fenchel duality concept [26, p.188] generalized in [27] onto the case of unbounded linear mappings in a Hilbert space. We note that this approach does not require that L has a bounded inverse or pseudo-inverse. The latter is important for the treatment of high index DAEs which are known to be ill-posed in $\mathbb{L}_2$ in a sense that the range of the corresponding mapping L is not closed[3] and null space is not trivial (see for instance [17]). This leads to $\sup\{\cdot\} = +\infty$ in (1.1) for some ℓ and u. It turns out that $\sup\{\cdot\}$ is finite if and only if a DAE, adjoint to (1.2), is solvable for a given $\mathcal{L}(\cdot, u)$. The latter statement represents generalized Kalman Duality principle (see Theorem 2.2) for DAEs in the form (1.2), proving that the problem (1.1)- (1.2) is equivalent to a dual control problem with a strictly convex cost functional and constraint represented by an adjoint DAE. In the case of ellipsoidal $\mathscr{G}$, dual control problem reads as a Linear Quadratic (LQ) control problem for the adjoint DAE. We note that a minimax state estimation framework based on a less general version of duality principle (the case of ellipsoidal bounding set $\mathscr{G}$ and DAEs with regular coefficients) was presented in [28].

If $F = I$ then the dual control problem may be solved by classical optimality conditions (Pontryagin Maximum Principle or dynamic programming approach).

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[2] It was mentioned in [22] that in the context of control problems conversion of a DAE to ODE leads to technical restrictions which may be avoided if one treats DAE in its original form.

[3] Above we mentioned that the solution of DAE may depend on derivatives of the input f implying that the range of L is a non-closed linear set.

However, they may be inapplicable for this problem if $F \in \mathbb{R}^{m\times n}$ (see, for instance, example in subsection 2.2). Maximum principle was applied to DAEs in the form (1.2) with quadratic time-varying F in [22] where the authors investigate existence of the optimal control in a feed-back form and study solvability of the Riccati equation with algebraic constraints. In [20] the authors derive maximum principle for a class of strangeness free[4] DAEs by a regularizing linear feed-back. In this paper we propose a regularization approach allowing to derive approximations for optimal controls and corresponding trajectories of the adjoint DAE without restricting the structure of F and $C(t)$. The main idea behind it is to reinterpret the adjoint DAE as an operator equation with a linear closed mapping D. As it was mentioned above, the range of D is not necessary closed and so the corresponding operator equation may be ill-posed. Due to this we derive an optimality conditions using a regularization lemma of [27]. This lemma is based on the Tikhonov regularization approach [29] and allows to approximate generalized solutions of linear ill-posed operator equations. In particular, it allows to derive optimality conditions for LQ control problem with DAE constraints in the form of a two-point boundary value problem for a zero-index DAE[5]. Using this optimality conditions we derive a minimizing sequence $\{\hat{u}_\varepsilon\}$ for the functional (1.1). This sequence converges in $\mathbb{L}_2$ if the adjoint DAE is solvable and diverges[6] to $+\infty$ otherwise. In fact, in the latter case the sequence $\{\hat{u}_\varepsilon\}$ converges weakly to a linear combination of distributions (see discussion presented after the proof of Proposition 2.8 for details). We represent $\hat{u}_\varepsilon$ in the form of a feed-back control splitting DAE into differential and algebraic parts by means of Singular Value Decomposition (SVD) and deriving Ricatti equation on a subspace which correspond to the differential part of DAE. This allows us to consider an infinite horizon problem. SVD was applied in [30] in order to derive optimality conditions for LQ finite and infinite horizon optimal control problems for regular stationary DAEs. Necessary and sufficient optimality conditions for finite horizon LQ problems with stationary singular DAE constraints were presented in [3]. We conclude discussing the application of the problem (1.1) to minimax state estimation for DAEs. In particular, we show that the optimal control represents the minimax estimate of DAE's state vector and cost function in (1.1) describes the minimax estimation error.

This paper is organized as follows. Subsection 1.1 contains all notations, subsection 1.2 describes the formal problem statement. In section 2 we present main results: subsection 2.1 describes preliminary results (operator interpretation of DAE, duality lemma and regularization lemma), subsection 2.2 introduces a Dual Control Problem for a general case of convex $\mathscr{G}$ (Theorem 2.4) and for the case of ellipsoidal $\mathscr{G}$ (Corollary 2.6) and represents optimality conditions (Proposition 2.8). Subsection 2.3.2 discusses infinite horizon problem. Subsection 2.4 presents an application to state estimation. Section 3 contains conclusions.

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[4]Any DAE from the latter class may be converted into the DAE with a regular matrix pencil $sF - C$.

[5]Zero-index DAE may be converted into an equivalent ODE without differentiation of the input

[6]The $\mathbb{L}_2$-norms of u_ε diverge to $+\infty$.

1.1. Notation

$\mathbb{R}^n$ denotes the arithmetic n-dimensional Euclidean space; $\mathbb{L}^2(t_0,t_1,\mathbb{R}^n)$ denotes the space of square-integrable functions on (t_0,t_1) with values in $\mathbb{R}^n$ (in what follows we will often write $\mathbb{L}_2$ or $\mathbb{L}_2(t_0,t_1)$ referring $\mathbb{L}_2(t_0,t_1,\mathbb{R}^k)$ where the dimension k will be defined by the context); $\mathbb{H}^1(t_0,t_1,\mathbb{R}^n)$ denotes a space of absolutely continuous functions with $\mathbb{L}_2$-derivative and values in $\mathbb{R}^n$; $\mathbb{H}^{1,\mathrm{F}}(t_0,t_1)$ denotes a space of $\mathbb{L}_2$-functions $x(\cdot)$ such that $Fx(\cdot)\in\mathbb{H}^1(t_0,t_1,\mathbb{R}^m)$ for $F\in\mathbb{R}^{m\times n}$, $\mathbb{H}^{1,\mathrm{F}}_0 := \{x\in\mathbb{H}^{1,\mathrm{F}} : Fx(t_0)=0\}$; $\mathbb{C}(t_0,t_1,\mathbb{R}^m)$ denotes a space of continuous functions with values in $\mathbb{R}^m$; $f(\cdot)$ or f denotes a function (as an element of a functional space); $f(t)$ denotes the value of f; the prime $'$ denotes the operation of taking adjoint: L' denotes adjoint operator, F' denotes the transposed matrix; F^+ denotes the pseudoinverse matrix; I_n denotes $n\times n$-identity matrix; $0_{n\times m}$ denotes $n\times m$-zero matrix, $I_0:=0$; $\mathscr{R}(L)$, $\mathscr{N}(L)$ and $\mathscr{D}(L)$ denote the range, null-space and domain of the mapping L; x^Ty denotes the inner product of vectors $x,y\in\mathbb{R}^n$, $\|x\|^2:=x^Tx$; $\langle\cdot,\cdot\rangle$ denotes the inner product in a Hilbert space $\mathcal{H}$, $\|x\|^2_{\mathcal{H}}:=\langle x,x\rangle$; for $S\in\mathbb{R}^{n\times n}$ we write $S>0$ if $x^TSx>0\ \forall x\neq 0$; $c(G,\cdot)$ denotes the support function of a set G; $\delta(G,\cdot)$ denotes the indicator of G: $\delta(G,f)=0$ if $f\in G$ and $+\infty$ otherwise; $\operatorname{int} G$ denotes the interior of G; $\mathcal{U}(a,b)$ denotes the uniform distribution supported over $[a,b]$.

1.2. Problem statement

Consider the following differential-algebraic equation

$$\frac{d(Fx)}{dt} = C(t)x(t)+f(t),\quad Fx(t_0)=0\,. \tag{1.3}$$

We define solution of (1.3) as follows.

Definition 1.1. $x(\cdot)\in\mathbb{L}_2(t_0,t_1,\mathbb{R}^n)$ is said to be a weak solution of (1.3) if $x(\cdot)\in\mathbb{H}^{1,\mathrm{F}}_0$ and $\frac{d(Fx)}{dt}$ is almost everywhere (except a set of zero Lebesgue measure) equal to the right hand side of (1.3).

We further assume that $f\in\mathscr{G}$, where $\mathscr{G}$ is a given convex closed bounded subset of $\mathbb{L}_2$. Take $u(\cdot)\in\mathbb{L}_2(t_0,t_1,\mathbb{R}^p)$, $\ell\in\mathbb{R}^m$ and let us represent a linear functional $\mathcal{L}(\cdot,\cdot)$ in the following form[7]: $\mathcal{L}(x,u)=\ell^TFx(t_1)-\int_{t_0}^{t_1}u^THxdt$. We arrive to the following cost functional:

$$\mathscr{I}(u) := \int_{t_0}^{t_1} u^TRudt + \sup_{f\in\mathscr{G}}\{\ell^TFx(t_1)-\int_{t_0}^{t_1}u^THxdt\}^2\,, \tag{1.4}$$

where $R(\cdot),R^{-1}(\cdot)\in\mathbb{C}(t_0,t_1,\mathbb{R}^{p\times p})$, $R'(t)=R(t)>0$, $H(\cdot)\in\mathbb{C}(t_0,t_1,\mathbb{R}^{p\times n})$, $-\infty<t_0<t_1\le+\infty$. The term $\sup_{f\in\mathscr{G}}\{\cdot\}$ in (1.4) means that sup is taken over all functions $x(\cdot)$ and $f(\cdot)$ such that $f(\cdot)\in\mathscr{G}$ and $x(\cdot)$ solves DAE (1.3) in sense of definition 1.1.

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[7]Note that this functional is not bounded in $\mathbb{L}_2$ with respect to $x(\cdot)$

Consider the following convex optimization problem

$$\mathscr{I}(u) \to \inf_{u \in \mathbb{L}_2} := \mathscr{I}^* . \tag{1.5}$$

Our aim is 1) to derive a dual problem for (1.5) and 2) to construct a sequence $\{u_\varepsilon\}$ such that $\lim \mathscr{I}(u_\varepsilon) \to \mathscr{I}^*$.

2. Main results

This section contains derivation of the generalized Kalman duality principle for DAE (1.3) and its application to the construction of the minimization sequence.

2.1. Preliminaries

In this subsection we briefly mention some mathematical results which serve a basis for the proofs presented in the next subsection. To make the text more readable we will often drop the argument t of functions and matrices used below.

2.1.1. Integration by parts formula. Take x and w such that $Fx \in \mathbb{H}^1(t_0, t_1, \mathbb{R}^m)$ and $F'w \in \mathbb{H}^1(t_0, t_1, \mathbb{R}^n)$, and assume that $t_1 < +\infty$. It was proved in [32] that:

$$\begin{aligned} (F'w(t_1))^T F^+ Fx(t_1) - (F'w(t_0))^T F^+ Fx(t_0) \\ = \int_{t_0}^{t_1} (w^T \frac{d(Fx)}{dt} + x^T \frac{d(F'w)}{dt}) dt \,. \end{aligned} \tag{2.1}$$

Let us now consider the case $t_1 = +\infty$. We note that $Fx \in \mathbb{H}^1(t_0, +\infty, \mathbb{R}^m)$ implies $Fx(t_1) = \lim_{t \to t_1} Fx(t) = 0$. Taking this into account we derive from (2.1) the following formula:

$$-\int_{t_0}^{+\infty} x^T \frac{d(F'w)}{dt} dt = (F'w(t_0))^T F^+ Fx(t_0) + \int_{t_0}^{+\infty} w^T \frac{d(Fx)}{dt} dt \tag{2.2}$$

2.1.2. Operator interpretation for DAE. Let us give an operator interpretation for DAE (1.3). Assume $t_1 < +\infty$ and define

$$(Lx)(t) = \frac{d(Fx)}{dt} - C(t)x(t), \quad x(\cdot) \in \mathscr{D}(L) := \mathbb{H}_0^{1,\mathrm{F}} \,. \tag{2.3}$$

Operator L maps $\mathscr{D}(L)$ into the Hilbert space $\mathbb{L}_2$. By definition, (1.3) is equivalent to the operator equation $Lx = f$. It was proved in [32] that L is a closed dense defined linear mapping and its adjoint $L' : \mathscr{D}(L') \subset \mathbb{L}_2 \to \mathbb{L}_2$ is given by[8]:

$$\begin{aligned} &(L'z)(t) = -\frac{d(F'z)}{dt} - C'(t)z(t), \quad z(\cdot) \in \mathscr{D}(L'), \\ &\mathscr{D}(L') := \{z \in \mathbb{H}^{1,\mathrm{F}'} : F'z(t_1) = 0\} \,. \end{aligned} \tag{2.4}$$

[8]This fact follows easily from the integration by parts formula (2.1)

Remark 2.1. It is common in the engineering literature [30] to consider DAEs in the form $F\frac{dx}{dt} = C(t)x(t) + f(t)$. We stress that the linear operator $(\tilde{L}x)(t) = F\frac{dx}{dt} - C(t)x(t)$ with domain $\mathscr{D}(\tilde{L}) = \mathbb{H}^1(t_0, t_1, \mathbb{R}^n)$ is not closed and its closure coincides with L defined by (2.3). In other words, the graph of operator $G(\tilde{L}) = \{(x, \tilde{L}x), x \in \mathscr{D}(L)\}$ is not closed and $G(L)$ is a closure of $G(\tilde{L})$. From this point of view it is more convenient to work with DAE in the form (1.3). Further properties of L in the form (2.3) were discussed in [32].

2.1.3. Duality lemma. In what follows $\mathcal{H}_{1,2}$ are assumed to be Hilbert spaces. In order to compute $\sup\{\cdot\}$ in (1.4) we will use the following duality result.

Lemma 2.2. *Assume $L : \mathscr{D}(L) \subset \mathcal{H}_1 \to \mathcal{H}_2$ is a closed dense-defined linear mapping, $\mathscr{G} \subset \mathcal{H}_2$ is a closed bounded convex set and $\mathcal{F} \in \mathcal{H}_1$, $z \in \mathcal{H}_2$. Then*

$$\sup_{x \in \mathscr{D}(L)} \{\langle \mathcal{F}, x\rangle, Lx \in \mathscr{G}\} = \inf_{b \in \mathscr{D}(L')} \{c(\mathscr{G}, b), L'b = \mathcal{F}\}\,, \tag{2.5}$$

$$\sup_f \{\langle z, f\rangle, f \in \mathscr{R}(L) \cap \mathscr{G}\} = \inf_{v \in \mathscr{D}(L')} \{c(\mathscr{G}, z - v), L'v = 0\}. \tag{2.6}$$

Proof. The proof of (2.5)–(2.6) for the case of a bounded L was presented in [26, p.188]. It is based on the Young-Fenchel duality concept. Generalization to the case of an unbounded L may be found in [27]. □

2.1.4. Tikhonov regularization for DAEs. In order to derive optimality conditions we will apply the following lemma:

Lemma 2.3 (regularization lemma). *Assume $D : \mathscr{D}(D) \subset \mathcal{H}_1 \to \mathcal{H}_2$ is a closed dense-defined linear mapping and $g \in \mathcal{H}_2$. Then the projection problem*

$$\|Dx - g\|^2_{\mathcal{H}_2} + \varepsilon\|x\|^2_{\mathcal{H}_1} \to \min_{x \in \mathscr{D}(D)} \tag{2.7}$$

has a unique solution $x_\varepsilon \in \mathscr{D}(D)$ and x_ε satisfies a system of operator equations:

$$\begin{aligned} Dx + \varepsilon p &= g\,,\\ D'p &= x\,. \end{aligned} \tag{2.8}$$

If the projection $\tilde{g}$ of g onto the closure of $\mathscr{R}(D)$ belongs to $\mathscr{R}(D)$ then $\|x_\varepsilon - \hat{x}\|_{\mathcal{H}_1} \to 0$ where $\|\hat{x}\|_{\mathcal{H}_1} = \min\{\|x\|_{\mathcal{H}_1}, Dx = \tilde{g}\}$, otherwise $\|x_\varepsilon\|_{\mathcal{H}_1} \to +\infty$.

Proof of Lemma 2.3. The proof of the Lemma is a generalization of the well-known results from [29]. This generalization is based on the von Neumann graph method for closed linear mappings: the graph of the closed linear mapping is a closed linear subset. Further details of the proof are available in [32]. □

2.2. Kalman Duality principle

In this subsection for a generic convex bounded closed set $\mathscr{G}$ we convert the optimization problem (1.5) into a Dual Control Problem for adjoint DAE applying Lemma 2.2.

Theorem 2.4 (dual control problem). *Assume* $t_1 < +\infty$. *Then* $\mathscr{I}^* = \inf_{u\in\mathbb{L}_2}\mathscr{I}(u) < +\infty$ *if and only if*

$$\frac{d(F'z)}{dt} = -C'(t)z(t) + H'(t)u(t), \quad F'z(t_1) = F'\ell \tag{2.9}$$

for some $z(\cdot) \in \mathbb{H}^{1,\mathrm{F}'}$ *and* $u(\cdot) \in \mathbb{L}_2$. *If* (2.9) *has a solution then*

$$\mathscr{I}(u) = \min_v c^2(\mathscr{G}, z - v) + \int_{t_0}^{t_1} u^T R u dt, \tag{2.10}$$

provided $v(\cdot)$ *obeys* (2.9) *with* $u(\cdot) = 0$ *and* $\ell = 0$.

Proof. Take $\ell \in \mathbb{R}^m$, $u(\cdot) \in \mathbb{L}_2$. Let us transform $\mathcal{L}(x,u) := \ell^T F x(t_1) - \int_{t_0}^{t_1} u^T H x dt$. There exists[9] $w(\cdot) \in \mathbb{L}_2(t_0, t_1, \mathbb{R}^m)$ such that:

$$F'w(\cdot) \in \mathbb{H}^1(t_0, t_1, \mathbb{R}^n), \quad F'w(t_1) = F'\ell\,. \tag{2.11}$$

Noting that [33, p.36] $F = FF^+F$ we have by (2.11):

$$\ell^T F x(t_1) = (F'\ell)^T F^+ F x(t_1) = (F'w(t_1))^T F^+ F x(t_1)\,. \tag{2.12}$$

As $x(\cdot) \in \mathbb{H}_0^{1,\mathrm{F}}$ we can write using (2.1) that

$$\begin{aligned}(F'w(t_1))^T F^+ F x(t_1) &- (F'w(t_0))^T F^+ F x(t_0) \\ &= \int_{t_0}^{t_1} (w^T \frac{d(Fx)}{dt} + x^T \frac{d(F'w)}{dt}) dt \\ &\overset{\text{by (1.3)}}{=} \int_{t_0}^{t_1} w^T f dt + \int_{t_0}^{t_1} x^T (\frac{d(F'w)}{dt} + C'w) dt\,.\end{aligned} \tag{2.13}$$

By definition of $\mathcal{L}(x,u)$, (2.12) and (2.13) we get for $x \in \mathbb{H}_0^{1,\mathrm{F}}$:

$$\mathcal{L}(x,u) = \int_{t_0}^{t_1} w^T f dt + \int_{t_0}^{t_1} x^T (\frac{d(F'w)}{dt} + C'w - H'u) dt\,. \tag{2.14}$$

for some $f(\cdot) \in \mathscr{G}$ such that (1.3) has a solution. In other words $f(\cdot) \in \mathscr{R}(L)$ with L defined by (2.3). As $\mathscr{G}$ is bounded we get that

$$\sup_{f\in\mathscr{G}} \mathcal{L}(x,u) < +\infty \Leftrightarrow \sup_{f\in\mathscr{G}} \{\int_{t_0}^{t_1} x^T (\frac{d(F'w)}{dt} + C'w - H'u) dt\} < +\infty\,. \tag{2.15}$$

From the definition (1.4) we deduce from (2.15) that

$$\mathscr{I}^* < +\infty \Leftrightarrow \sup_{f\in\mathscr{G}} \{\int_{t_0}^{t_1} x^T (\frac{d(F'w)}{dt} + C'w - H'u) dt\} < +\infty\,. \tag{2.16}$$

Now assume that $\mathscr{I}^* < +\infty$ for the given ℓ and $u(\cdot)$. We claim that there exists $z(\cdot)$ such that (2.9) holds for these ℓ and $u(\cdot)$. To see this let us define $\mathcal{F} := \frac{d(F'w)}{dt} + C'w - H'u$. It is clear that $\mathcal{F} \in \mathbb{L}_2$. Recalling (2.3) we note that sup

[9] For instance, $w(t) \equiv \ell$.

in (2.16) is, in fact, taken over all $x(\cdot) \in \mathscr{D}(L)$ such that $Lx \in \mathscr{G}$. Thus (2.16) implies:

$$\sup_{x\in\mathscr{D}(L)} \{\langle \mathcal{F}, x\rangle, Lx \in \mathscr{G}\} < +\infty\,.$$

On the other hand, by Lemma 2.2:

$$\sup_{x\in\mathscr{D}(L)} \{\langle \mathcal{F}, x\rangle, Lx \in \mathscr{G}\} = \inf_{b\in\mathscr{D}(L')} \{c(\mathscr{G}, b), L'b = \mathcal{F}\} < +\infty\,. \tag{2.17}$$

By (2.17) there exists at least one $b \in \mathscr{D}(L')$ such that $L'b = \mathcal{F}$ or (using (2.4)):

$$-\frac{d(F'b)}{dt} - C'b = \frac{d(F'w)}{dt} + C'w - H'u\,.$$

Setting $z := (w + b)$ we obtain $\frac{d(F'z)}{dt} + C'z - H'u = 0$ and $F'z(t_1) = F'\ell$. This proves (2.9) has a solution in $\mathbb{H}^{1,\mathrm{F}'}$.

Now assume that $z(\cdot)$ solve (2.9) for a given ℓ and $u(\cdot)$. Let us prove that $\mathscr{I}^* < +\infty$. As $z(\cdot)$ verifies (2.11) we can apply (2.14) with $w = z$. We get

$$\mathcal{L}(x, u) = \int_{t_0}^{t_1} z^T f dt\,, f \in \mathscr{G}_1 := \mathscr{G} \cap \mathscr{R}(L)$$

with L defined by (2.3). This and (2.6) allows us to write:

$$\sup_{f\in\mathscr{G}} \{\mathcal{L}(x, u)\}^2 = \sup_{f\in\mathscr{G}_1} \{\int_{t_0}^{t_1} z^T f dt\}^2 = \inf_{v\in\mathscr{N}(L')} \{c(\mathscr{G}, z - v)\}^2\,. \tag{2.18}$$

As $\mathscr{G}_1$ is a bounded convex set, (2.18) implies that $\mathscr{I}^* < +\infty$.

Let us finally prove (2.10). Recalling the definition of L' we see that $v(\cdot) \in \mathscr{N}(L')$ if and only if $v(\cdot)$ solves (2.9) with $u = 0$ and $\ell = 0$. Thus

$$\inf_{v\in\mathscr{N}(L')} \{c(\mathscr{G}, z - v)\}^2 = \min_v c^2(\mathscr{G}, z - v)\,,$$

provided (2.9) has a solution. Combining (1.4), (2.18) and the latter formula we derive (2.10). This completes the proof. □

Remark 2.5. We stress that the proof of Theorem 2.4 is based on the following geometric idea: given u and ℓ to find a maximum of the linear functional

$$\mathcal{L}(x, u) = \ell^T F x(t_1) - \int_{t_0}^{t_1} u^T H x dt$$

over the set $\{x \in \mathscr{D}(L) : Lx \in \mathscr{G}\}$. The main difficulty here is that $\mathcal{L}(x, u)$ is not a linear continuous functional so it can not be represented by means of an inner product in $\mathbb{L}_2$ and, thus, the direct application of Lemma 2.2 is not possible. To overcome this we "shift" the state space of the adjoint DAE considering the auxiliary function w (see (2.14)) and use integration by parts.

Corollary 2.6 (quadratic cost). *Assume that the conditions of Theorem 2.4 are fulfilled and*

$$\mathscr{G} := \{f(\cdot) : \int_{t_0}^{t_1} f^T(t)Q(t)f(t)dt \leq 1\}, \tag{2.19}$$

where $Q^{-1}(\cdot), Q(\cdot) \in \mathbb{C}(t_0, t_1, \mathbb{R}^{m\times m})$*,* $Q(t) = Q'(t) > 0$*. Then minimax control problem* (1.5) *is equivalent to the following linear-quadratic control problem with DAE constraint:*

$$\begin{aligned} &N(z,u) := \int_{t_0}^{t_1} z^T Q^{-1} z dt + \int_{t_0}^{t_1} u^T R u dt \to \min_{(z,u)}, \\ &\frac{d(F'z)}{dt} = -C'(t)z(t) + H'(t)u(t), \quad F'z(t_1) = F'\ell\,. \end{aligned} \tag{2.20}$$

Proof. It is easy to check that the support function of the ellipsoid (2.19) is given by the following expression:

$$c^2(\mathscr{G}, b(\cdot)) = \int_{t_0}^{t_1} b^T Q^{-1} b dt\,.$$

Hence, by (2.10)

$$\mathscr{I}(u) = \min_v \{\int_{t_0}^{t_1} (z-v)^T Q^{-1}(z-v)dt\} + \int_{t_0}^{t_1} u^T R u dt\,,$$

provided $v(\cdot)$ satisfies (2.9) with $u(\cdot) = 0$ and $\ell = 0$. If $u(\cdot)$, $z(\cdot)$ solve (2.9) then $u(\cdot)$ and $z(\cdot) - v(\cdot)$ also solve (2.9) so that

$$\min_v \{\int_{t_0}^{t_1} (z-v)^T Q^{-1}(z-v)dt\} = \min_z \{\int_{t_0}^{t_1} z^T Q^{-1} z dt\}\,,$$

where $z(\cdot)$ runs through the set of all solutions of (2.9) corresponding to $u(\cdot)$ and ℓ. Therefore, $\min_u \mathscr{I}(u) = \min_{u,z}\{N(u,z), (u(\cdot), z(\cdot)) \text{ solve } (2.9)\}$. This completes the proof. □

2.3. Optimality conditions

In this subsection we apply Lemma 2.3 in order to derive optimality conditions for the dual control problem in the form of Pontryagin maximum principle, provided $\mathscr{G}$ has an "ellipsoidal" shape (2.19). We construct a minimizing sequence $\{u_\varepsilon\}$ and represent each u_ε in the form of the feed-back control using Ricatti equation on a subspace[10].

[10]This subspace coincides with a range of the matrix F' and represents a differentiable part of the DAE solution x.

2.3.1. Finite horizon problem. In what follows we will present optimality conditions for the control problem (2.20). If $F = I$ then $\hat{u} = R^{-1}Hp$ solves (2.20), provided $p(\cdot)$ verifies the following optimality conditions (Euler-Lagrange System in the Hamilton form [26]):

$$\begin{aligned} &\frac{dFp}{dt} = Cp + Q^{-1}z,\ Fp(t_0) = 0, \\ &\frac{dF'z}{dt} = -C'z + H'R^{-1}Hp,\ F'z(t_1) = F'\ell\,. \end{aligned} \tag{2.21}$$

In the general case $F \in \mathbb{R}^{m\times n}$, let us assume that the system (2.21) is solvable. One can prove using direct variational method (see [26])) that $\hat{u} = R^{-1}Hp$ solves (2.20). Although this assumption allows one to solve the optimal control problem (2.20), it may be too restrictive. To illustrate this, let us consider an example. Define

$$F = \left[\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{smallmatrix}\right], C = \left[\begin{smallmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & -1 \end{smallmatrix}\right], H = \left[\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{smallmatrix}\right] \tag{2.22}$$

and take $Q(t) = I_2$, $R(t) = I_3$. In this case (2.21) reads as:

$$\begin{aligned} &\frac{dz_1}{dt} = z_2 + p_1, \quad z_1(t_1) = \ell_1, \quad -z_1 \equiv 0, \\ &\frac{dz_2}{dt} = p_2, \quad z_2(t_1) = \ell_2, \quad z_2 + p_4 = 0, \\ &\frac{dp_1}{dt} = p_3 + z_1, \quad p_1(t_0) = z_1(t_0), \\ &\frac{dp_2}{dt} = -p_1 - p_4 + z_2, \quad p_2(t_0) = z_2(t_0)\,. \end{aligned} \tag{2.23}$$

We claim that (2.23) has a solution iff $\ell_1 = \ell_2 = 0$. Indeed, $z_1(t) \equiv 0$ implies $z_1(t_1) = \ell_1 = 0$, $-z_2 = p_1 = p_4$ and $\frac{d}{dt}p_1 = p_3$. According to this we rewrite (2.23) as follows:

$$\frac{dp_1}{dt} = -p_2\,, \frac{dp_2}{dt} = -3p_1\,, p_2(t_0) = 0\,, p_1(t_0) = 0\,, p_1(t_1) = \ell_2, \tag{2.24}$$

It is clear that (2.24) has a solution iff $\ell_2 = 0$. Thus, the classical optimality conditions (2.21) catch just the trivial optimal control $\hat{u} = 0$ and put an artificial constraint onto ℓ, restricting the class of cost functions (1.4). On the other hand, the control problem (2.20) may have non-trivial solution $\hat{u} = (\hat{u}_1, \hat{u}_2, \hat{u}_3)^T$, provided $\ell_1 = 0$ and $\ell_2 \neq 0$. Indeed, consider the following control problem:

$$N(z,u) = \sum_{i=1}^{2} \|z_i\|^2_{\mathbb{L}_2} + \sum_{j=1}^{3} \|u_j\|^2_{\mathbb{L}_2} \to \min_{u_i, z_i}\,, \tag{2.25}$$

$$\frac{dz_1}{dt} - z_2 - u_1 = 0, \quad z_1(t_1) = \ell_1, \quad -z_1 = 0\,, \tag{2.26}$$

$$\frac{dz_2}{dt} - u_3 = 0, \quad z_2(t_1) = \ell_2, \quad -z_2 - u_2 = 0\,. \tag{2.27}$$

If $\hat u_{1,2}$ solves (2.26)-(2.27) then $\hat u_{1,2} = -z_2$. Hence, $\hat u_3$ solves the following problem:

$$\min_u\{\int_{t_0}^{t_1} 3z_2^2 + u_3^2 dt\,, \frac{dz_2}{dt} = u_3,\,, z_2(t_1) = \ell_2\}\,. \tag{2.28}$$

The optimality condition takes the classical form:

$$\frac{dz_2}{dt} = \hat u_3\,, \frac{dp}{dt} = 3z_2\,, \hat u_3 = p\,, z_2(t_1) = \ell_2\,, p(t_0) = 0\,. \tag{2.29}$$

Remark 2.7. The situation described so far has a nice operator interpretation in the spirit of Lemma 2.3. Let us define

$$\begin{aligned} &D(z,u)(t) := \big[-\frac{d(F'z)}{dt} - C'(t)z(t) + H'(t)u(t), F'z(t_1)\big],\ (z,u) \in \mathscr{D}(D),\\ &\mathscr{D}(D) = \mathbb{H}^{1,\mathrm{F}'} \times \mathbb{L}_2(t_0,t_1,\mathbb{R}^p)\,. \end{aligned} \tag{2.30}$$

Then (2.20) may be interpreted as follows: to find a solution $(\hat u, \hat z)$ of the operator equation $D(z,u) = (0, F'\ell)$ with the minimal Euclidean norm $\|(z,u)\|^2 := N(z,u)$. It is well-known from the general operator theory (see [23, p.14]) that $(\hat u, \hat z) \in \overline{\mathscr{R}(D')}$. If $\mathscr{R}(D)$ is a closed set then $D'\hat p = (\hat u, \hat z)$ for some $\hat p \in \mathscr{D}(D')$ so that the system of operator equations $D(z,u) = (0, F'\ell), \quad D'p = (z,u)$, has a solution $(\hat u, \hat z, \hat p)$, provided $(0, F'\ell) \in \mathscr{R}(D)$. It turns out that (see proof of Proposition 2.8 below) the latter system is equal to the Euler-Lagrange equation (2.21). If $\mathscr{R}(D)$ is not closed then (2.21) is not necessary solvable even for $(0, F'\ell) \in \mathscr{R}(D)$ and the example above illustrates this. The discussion above suggests a way to find a solution for the ill-posed linear quadratic control problem (2.20): the idea is to apply Tikhonov regularization approach [29] in order to construct a minimizing sequence $\{u_\varepsilon\}$.

Proposition 2.8 (optimality conditions). *The DAE boundary-value problem*

$$\begin{aligned} &\frac{d(F'z)}{dt} = -C'z + H'\hat u + \varepsilon p, \quad F'z(t_1) + \varepsilon F^+Fp(t_1) = F'\ell,\\ &\frac{d(Fp)}{dt} = Cp + Q^{-1}z, \quad \hat u = R^{-1}Hp, \quad Fp(t_0) = 0 \end{aligned} \tag{2.31}$$

has a unique solution $\hat u_\varepsilon$, $\hat p_\varepsilon$, $\hat z_\varepsilon$ for any $\varepsilon > 0$. If (2.9) has a solution then there exists $\hat u$ and $\hat z$ such that 1) $\hat u_\varepsilon \to \hat u$, $\hat z_\varepsilon \to \hat z$ in $\mathbb{L}_2$, and 2) $\hat u$ and $\hat z$ verify (2.9) and

$$N(\hat u_\varepsilon, \hat z_\varepsilon) \to N(\hat u, \hat z) = \inf_{(u,z)\ solves\ (2.9)} N(u,z) = \mathscr{I}^*\,. \tag{2.32}$$

Proof. Let us recall the definition of the operator D given in (2.30). As it was proved in [32], D is a closed dense defined linear mapping and the adjoint D' is defined as follows:

$$\begin{aligned} &D'(p,p_0)(t) = \big[\begin{smallmatrix} \frac{d(Fp)}{dt} - C(t)p(t) \\ H(t)p(t) \end{smallmatrix}\big], \quad (p,p_0) \in \mathscr{D}(D')\,,\\ &\mathscr{D}(D') = \{(p,p_0) : p(\cdot) \in \mathbb{H}_0^{1,\mathrm{F}}, \quad p_0 = F^+Fp(t_1) + d, Fd = 0\}\,. \end{aligned} \tag{2.33}$$

Let us define Tikhonov function

$$\mathcal{T}_\varepsilon(u,z) = \|D(z,u) - (0, F'\ell)\|^2_{\mathbb{L}_2\times\mathbb{R}^n} + \varepsilon N(u,z)\,.$$

By Lemma 2.3 we get that the projection problem $\mathcal{T}_\varepsilon(u,z) \to \inf_{(u,z)\in\mathscr{D}(D)}$ has a unique solution $(\hat u_\varepsilon, \hat z_\varepsilon)$ and $(\hat u_\varepsilon, \hat z_\varepsilon)$ solves the following Euler-Lagrange equation:

$$\begin{aligned} &D(z,u) + \varepsilon(p,p_0) = (0, F'\ell)\,,\\ &D'(p,p_0) = (Q^{-1}z, R^{-1}u)\,. \end{aligned} \tag{2.34}$$

Substituting (2.30), (2.33) into (2.34) one gets (2.31). Thus (2.31) has a unique solution $(\hat u_\varepsilon, \hat z_\varepsilon, \hat p_\varepsilon)$. If (2.9) has a solution then $(0, F'\ell) \in \mathscr{R}(D)$ so that, by Lemma 2.3, there exists $(\hat u, \hat z)$ such that $\|u_\varepsilon - \hat u\|^2_{\mathbb{L}_2} + \|z_\varepsilon - \hat z\|^2_{\mathbb{L}_2} \to 0$, $N(\hat u_\varepsilon, \hat z_\varepsilon) \to N(\hat u, \hat z)$ and $D(\hat z, \hat u) = (0, F'\ell)$. Thus $(\hat z, \hat u)$ solves (2.9) (by definition of D), verifies 1) and 2) and (2.32). □

Remark 2.9. Note, that for stationary $C(t)$ and $H(t)$, one can (see for instance [3]) give an efficient description of the set of all ℓ such that (2.9) is solvable. Having established the solvability, one can use the sequence $\{\hat u_\varepsilon, \hat z_\varepsilon\}$ in order to approximate the optimal control in (2.20) and the corresponding solution of (2.9). On the other hand, convergence of the minimizing sequence $\{\hat u_\varepsilon\}$ is equivalent (see [32]) to the solvability of (2.9).

Lemma 2.10. (2.31) *is equivalent to a boundary value problem for an ODE.*

Proof. The proof is by construction. Define $r := \mathrm{rang} F$ and $\Lambda = \mathrm{diag}(\lambda_1 \dots \lambda_r)$ where λ_i, $i = \overline{1,r}$ are positive eigen values of FF'. If $r = 0$ then (2.31) becomes trivial. Assume $r > 0$. It is easy to see, applying SVD decomposition [33] to F, that $F = U'SV$ where $UU' = I_m$, $V'V = I_n$ and $S = \left[\begin{smallmatrix} \Lambda^{\frac12} & 0_{r\times n-r} \\ 0_{m-r\times r} & 0_{m-r\times n-r}\end{smallmatrix}\right]$. Thus, multiplying the first equation of (2.31) by U, the second – by V, and changing variables one can reduce the general case to the case $F = \left[\begin{smallmatrix} I_r & 0_{r\times n-r} \\ 0_{m-r\times r} & 0_{m-r\times n-r}\end{smallmatrix}\right]$. In what follows, therefore, we can focus on this case only.

Having in mind the above 4-block representation for F we split vector $\ell = \left(\begin{smallmatrix}\ell_1\\ \ell_2\end{smallmatrix}\right)$ and the coefficients of (2.31) as follows: $C(t) = \left[\begin{smallmatrix} C_1 & C_2 \\ C_3 & C_4\end{smallmatrix}\right]$, $Q^{-1} = \left[\begin{smallmatrix} Q_1 & Q_2 \\ Q_2' & Q_4\end{smallmatrix}\right]$, $H'RH = \left[\begin{smallmatrix} S_1 & S_2 \\ S_2' & S_4\end{smallmatrix}\right]$. If 1) $n = r$ and $m > r$ we set $C_2 := 0_{r\times 1}$, $C_4 := 0_{m-r\times 1}$ and $S_2 := 0_{r\times 1}$, $S_4 := 0$; if 2) $n > r$ and $m = r$ we set $C_3 := 0_{1\times r}$, $C_4 := 0_{1\times n-r}$ and $Q_2 := 0_{m\times 1}$, $Q_4 := 0$; if 3) $n = m = r$ we set $C_4 := 0$, $C_2 := 0_{r\times 1}$, $C_3 := 0_{1\times r}$ and let S_i, Q_i be defined as in 1) and 2) respectively, $i \in \{2,4\}$. According to this (2.31) splits into a differential part:

$$\begin{aligned} &\frac{dp_1}{dt} = C_1p_1 + C_2p_2 + Q_1z_1 + Q_2z_2,\\ &\frac{dz_1}{dt} = -C_1'z_1 - C_3'z_2 + \varepsilon p_1 + S_1p_1 + S_2p_2,\\ &z_1(t_1) + \varepsilon p_1(t_1) = \ell_1, \quad p_1(t_0) = 0 \end{aligned} \tag{2.35}$$

and algebraic part:

$$
\begin{aligned}
&C_3p_1 + C_4p_2 + Q_2'z_1 + Q_4z_2 = 0,\\
&-C_2'z_1 - C_4'z_2 + S_2'p_1 + (\varepsilon I_{n-r} + S_4)p_2 = 0.
\end{aligned}
$$

Let us define

$$
\begin{aligned}
&W_\varepsilon(t) = \varepsilon I_{n-r} + S_4(t) + C_4'(t)Q_4^+(t)C_4(t)\,, M_\varepsilon(t) = W_\varepsilon^+(t)\,,\\
&D(t) = C_3'(t)Q_4^+(t)C_4(t) + S_2(t),\; B(t) = C_2'(t) - C_4'(t)Q_4^+(t)Q_2'(t)\,,\\
&A_\varepsilon(t) = C_1(t) - Q_2(t)Q_4^+(t)C_3(t) - B'(t)M_\varepsilon(t)D'(t)\,,\\
&Q_\varepsilon(t) = \varepsilon I_r - D(t)M_\varepsilon(t)D'(t) + S_1(t) + C_3'(t)Q_4^+(t)C_3(t)\,,\\
&S_\varepsilon(t) = Q_1(t) - Q_2(t)Q_4^+(t)Q_2'(t) + B'(t)M_\varepsilon(t)B(t)\,.
\end{aligned}
$$

Solving the algebraic equations for z_2, p_2 we find:

$$
\begin{aligned}
&Q_4z_2 = (-Q_2' - C_4M_\varepsilon B)z_1 + (C_4M_\varepsilon D' - C_3)p_1,\\
&p_2 = M_\varepsilon Bz_1 - M_\varepsilon D'p_1,
\end{aligned}
\tag{2.36}
$$

Substituting (2.36) into (2.35) one finds that p_1, z_1 solve the following two-point boundary value problem:

$$
\begin{aligned}
&\frac{dz_1}{dt} = -A_\varepsilon'z_1 + Q_\varepsilon p_1, \quad z_1(t_1) + \varepsilon p_1(t_1) = \ell_1,\\
&\frac{dp_1}{dt} = A_\varepsilon p_1 + S_\varepsilon z_1, \quad p_1(t_0) = 0\,.
\end{aligned}
\tag{2.37}
$$

(2.37) is uniquely solvable as it is equivalent to (2.31) and (2.31) has a unique solution by Proposition 2.8. □

Feed-back and dual representation for the sub-optimal control. The following corollary presents feed-back and dual forms for $\hat u_\varepsilon$ and $\hat u$. In addition, it provides the optimal and sub-optimal cost values. Define $\Phi_\varepsilon := \left[\begin{smallmatrix} K \\ M_\varepsilon(B - D'K)\end{smallmatrix}\right]$ and let q_ε, K_ε and $\hat x_\varepsilon^j$ solve the following differential equations:

$$
\frac{dK}{dt} = A_\varepsilon K + KA_\varepsilon' + S_\varepsilon - KQ_\varepsilon K, \quad K(t_0) = 0\,, \tag{2.38}
$$

$$
\frac{dq}{dt} = (-A_\varepsilon' + Q_\varepsilon K_\varepsilon)q, \quad q(t_1) = (I_r + \varepsilon K_\varepsilon(t_1))^{-1}\ell_1\,, \tag{2.39}
$$

$$
\frac{d\hat x}{dt} = (A_\varepsilon - K_\varepsilon Q_\varepsilon)\hat x + \Phi_\varepsilon' H'R^{-1}y_j(t), \quad \hat x(t_0) = 0\,. \tag{2.40}
$$

Corollary 2.11. *Assume that $\{y_j\}$ is a total system in $\mathbb{L}_2(t_0,+\infty)$. The sub-optimal control $\hat u_\varepsilon$ and optimal control $\hat u$ admit the following representations:*

$$\hat u_\varepsilon(t) = R^{-1}(t)H(t)\Phi_\varepsilon(t)q_\varepsilon(t)\,, \tag{2.41}$$

$$\hat u_\varepsilon(t) = \sum_{j\in\mathbb{N}} \ell_1^T(I_r+\varepsilon K_\varepsilon(t_1))^{-1}\hat x_\varepsilon^j(t_1)y_j(t)\mathbb{1}(t_0,t_1,t)\,, \tag{2.42}$$

$$\hat u(t) = \sum_{j\in\mathbb{N}} \inf_{\varepsilon>0}\{\ell_1^T(I_r+\varepsilon K_\varepsilon(t_1))^{-1}\hat x_\varepsilon^j(t_1)\}y_j(t)\mathbb{1}(t_0,t_1,t)\,. \tag{2.43}$$

Sub-optimal value of the cost function is given by:

$$\mathscr{I}(\hat u_\varepsilon) = N(\hat u_\varepsilon,\hat z_\varepsilon) = \ell^T F\hat p_\varepsilon(t_1) - \varepsilon(\|F^+F\hat p_\varepsilon(t_1)\|^2 + \|\hat p_\varepsilon\|^2_{\mathbb{L}_2})\,. \tag{2.44}$$

The optimal value may be represented as follows:

$$\mathscr{I}^* = \mathscr{I}(\hat u) = \inf_{\varepsilon>0} \ell_1^T(I_r+\varepsilon K_\varepsilon(t_1))^{-1}K_\varepsilon(t_1)(I_r+\varepsilon K_\varepsilon(t_1))^{-1}\ell_1\,. \tag{2.45}$$

Proof. We will split boundary-value problem (2.37) into Cauchy problems for proving (2.41). Let us introduce a matrix-valued function K such that $p_1 = Kz_1$. Differentiating the latter equality and using (2.37) we get that K solves (2.38) and $z_1 = q_\varepsilon$. Let $\hat p_\varepsilon$ solve (2.31). We can split $\hat p_\varepsilon$ as follows: $\hat p_\varepsilon = (p_1,p_2)^T$ where p_1 solves (2.37) and p_2 is defined by (2.36). Now, recalling that $p_1 = K_\varepsilon q_\varepsilon$ and $\hat u_\varepsilon = R^{-1}Hp_\varepsilon$ we obtain (2.41). To prove (2.42) we first note that[11] one can choose a total system $\{y_j\}$ in $\mathbb{L}_2(t_0,+\infty)$ so that $\{\mathbb{1}(t_0,t_1,\cdot)y_j\}$ would be total in $\mathbb{L}_2(t_0,t_1)$. Now, we take y_j and use (2.38)-(2.41) to get:

$$\int_{t_0}^{t_1} y_j^T(t)\hat u_\varepsilon(t)dt = \ell_1^T(I_r+\varepsilon K_\varepsilon(t_1))^{-1}\hat x_\varepsilon^j(t_1)\,. \tag{2.46}$$

In fact, right hand side of (2.46) represents the projection of $\hat u_\varepsilon$ onto j-basis function y_j. As the system $\{y_j\}$ is total we obtain (2.42). Now, by (2.32) and (2.46):

$$\inf_{\varepsilon\downarrow 0}\ell_1^T(I_r+\varepsilon K_\varepsilon(t_1))^{-1}\hat x_\varepsilon^j(t_1) = \lim_{\varepsilon\downarrow 0}\int_{t_0}^{t_1} y_j^T(t)\hat u_\varepsilon(t)dt = \int_{t_0}^{t_1} y_j^T(t)\hat u(t)dt\,,$$

so that (2.43) holds true. (2.44) can be easily proved using (2.31). Let us prove (2.45). Assume $\hat u$ and $\hat z$ are defined as in Proposition 2.8. Using sub-gradient inequality and (2.31) we find that $N(\hat u,\hat z) - N(\hat u_\varepsilon,\hat z_\varepsilon) \ge \varepsilon(\|F^+F\hat p_\varepsilon(t_1)\|^2 + \|\hat p_\varepsilon\|^2_{\mathbb{L}_2})$. Thus, by (2.32) we obtain:

$$\lim_{\varepsilon\to 0}\varepsilon(\|F^+F\hat p_\varepsilon(t_1)\|^2 + \|\hat p_\varepsilon\|^2_{\mathbb{L}_2}) = 0\,. \tag{2.47}$$

We note that $F\hat p_\varepsilon(t_1) = p_1 = K_\varepsilon(t_1)q_\varepsilon(t_1)$. Combining this with (2.39) we get:

$$\begin{aligned}\mathscr{I}(\hat u_\varepsilon) + \varepsilon\|\hat p_\varepsilon\|^2_{\mathbb{L}_2} &= \ell^T F\hat p_\varepsilon(t_1) - \varepsilon\|F^+F\hat p_\varepsilon(t_1)\|^2 \\ &= \ell_1^T(I_r+\varepsilon K_\varepsilon(t_1))^{-1}K_\varepsilon(t_1)(I_r+\varepsilon K_\varepsilon(t_1))^{-1}\ell_1\,.\end{aligned} \tag{2.48}$$

Now, (2.45) follows from (2.32), (2.47) and (2.48). This completes the proof. □

[11] For instance, $\exp(-t), \exp(-t)t, \exp(-t)t^2\ldots$ is total in $\mathbb{L}_2(t_0,t_1)$ for any $t_0 < t_1 \le +\infty$.

Remark 2.12. We stress that $\lim_{\varepsilon\downarrow 0} q_\varepsilon(t_1) = \ell_1$ by (2.37) and (2.47). This and (2.39) imply that $\lim_{\varepsilon\downarrow 0}(I_r + \varepsilon K_\varepsilon(t_1))^{-1}\ell_1 = \ell_1$.

2.3.2. Infinite horizon problem. In this subsection we study the case $t_1 = +\infty$. In the next corollary we approach the problem using duality principle 2.4 that leads to $\hat{u} = 0$. This fact fully agrees with a state estimation interpretation of the original problem (1.5) (see Section 2.4).

Corollary 2.13. *Assume $t_1 = +\infty$ and $x(\cdot) \in \mathbb{H}_0^{1,\mathrm{F}}(t_0, +\infty)$. Then, the problem (1.5) has the trivial solution $\hat{u} = 0$ only.*

Proof. We note that the definition of L given in (2.3) does not change and formula (2.2) shows that the adjoint of L is also defined by (2.4). Dual control problem modifies as follows. In the case $t_1 = +\infty$ we have that $Fx(t_1) = 0$ for any $x \in \mathbb{H}_0^{1,\mathrm{F}}$ by definition of $\mathbb{H}^1(t_0, +\infty)$ and so $\mathcal{L}(x, u) = \int_{t_0}^{+\infty} x^T(-H'u)dt$ for any $\ell \in \mathbb{R}^m$. Thus, we can assume, without loss of generality, that $\ell = 0$ for $t_1 = +\infty$. But then $\mathcal{L}(x, u)$ is a linear continuous functional, provided $u \in \mathbb{L}_2(t_0, +\infty)$, and so Lemma 2.2 is applicable. Using (2.5) on can easily prove the following assertion: if $t_1 = +\infty$ then $\mathscr{I}^* < +\infty$ if and only if there exists $u \in \mathbb{L}_2(t_0, +\infty)$ and $z \in \mathbb{H}^{1,\mathrm{F}'}(t_0, +\infty)$ such that

$$\frac{d(F'z)}{dt} = -C'(t)z(t) + H'(t)u(t) \tag{2.49}$$

and in this case

$$\mathscr{I}(u) = \min_v c^2(\mathscr{G}, z - v) + \int_{t_0}^{+\infty} u^T R u dt, \tag{2.50}$$

provided $v(\cdot)$ obeys (2.9) with $u(\cdot) = 0$. If $\mathscr{G}$ is defined by (2.19) then (2.50) is equivalent to the following linear-quadratic infinite horizon control problem:

$$\begin{aligned} N(z, u) &:= \int_{t_0}^{+\infty} z^T Q^{-1} z dt + \int_{t_0}^{+\infty} u^T R u dt \to \min_{(z,u)}, \\ \frac{d(F'z)}{dt} &= -C'(t)z(t) + H'(t)u(t) \end{aligned} \tag{2.51}$$

which obviously has a trivial solution $\hat{u} = 0$, $\hat{z} = 0$ only. □

Consider now the case $x \in \mathbb{H}_0^{1,\mathrm{F}}(t_0, t_1)$ for any $t_0 < t_1 < +\infty$. In what follows we will emphasise the dependence on the final time t_1 writing $\hat{u}_\varepsilon(\cdot, t_1)$, $\mathscr{I}(\cdot, t_1)$, $\mathscr{G}(t_1)$ and $\mathscr{I}^*(t_1)$.

Corollary 2.14. *Let $\hat{u}(\cdot, t_k)$ be defined by (2.43) and assume that:*

1) *$\exists\, \ell \in \mathbb{R}^m$ such that $\mathscr{I}^*(t_1) < +\infty$ for any $t_1 > t_0$;*
2) *$\exists\, \varepsilon_0 > 0$ such that $\hat{\sigma} := \sup_{t_1 > t_0} \ell_1^T K_{\varepsilon_0}(t_1)\ell_1 < +\infty$.*

Then there exists a sequence of optimal controls $\{\hat{u}(\cdot, t_k)\}_{k\in\mathbb{N}}$ such that:

$$\mathscr{I}^*(\infty) := \sup_{t_1 > t_0} \mathscr{I}(\hat{u}(\cdot, t_1), t_1) \le \sup_{t_1 > t_0} \ell_1^T K_\varepsilon(t_1)\ell_1\,. \tag{2.52}$$

Proof. Take an increasing sequence $\{t_s\}$ such that $\lim_{s\to\infty} t_s = +\infty$. Assumption 1) and (2.45) imply that for any $s \in \mathbb{N}$:

$$\begin{aligned}\mathscr{I}(\hat{u}(\cdot,t_s),t_s) &\le \ell_1^T (I_r + \varepsilon_0 K_{\varepsilon_0}(t_s))^{-1} K_{\varepsilon_0}(t_s)(I_r + \varepsilon_0 K_{\varepsilon_0}(t_s))^{-1}\ell_1 \\ &\le \ell_1^T K_{\varepsilon_0}(t_s)\ell_1 \le \hat{\sigma}\,.\end{aligned} \tag{2.53}$$

This proves (2.52). □

2.4. Example: application to state estimation

In this subsection we apply problem (1.5) to construct a minimax state estimate for a singular DAE. Namely, we will be looking for an estimate of a linear functional

$$\ell(x) := \ell^T F x(t_1), \quad x(\cdot) \text{ solves (1.3) for some } f(\cdot) \in \mathscr{G}$$

in the class of linear functionals $u(y) := \int_{t_0}^{t_1} u^T(t)y(t)dt$ defined on the so called "observed data" $y(t) = Hx(t) + \eta(t)$. Assuming that η is a realization of a random process such that $E\eta(t) = 0$ and $E\int_{t_0}^{t_1} \eta^T R^{-1}(t)\eta dt \le 1$ one defines a worst-case mean-squared estimation error (see [3, 9] for details) $\sigma(t_1,\ell,u) := \sup_{f\in\mathscr{G},\eta} E(\ell(x) - u(y))^2$ and seeks $\hat{u}$ minimizing σ for a given ℓ. Easy computation[12] shows that $\sigma(t_1,\ell,u) = \mathscr{I}(u)$. Therefore, a minimizer $\hat{u}$ of $\mathscr{I}$ – the minimax estimate – is a linear estimate of $\ell(x)$ having minimal worst-case mean-squared estimation error – so called minimax error. By (2.46), $\ell(x) - \hat{u}_\varepsilon(y) = \ell_1^T(x(t_1) - \hat{x}_\varepsilon(t_1))$ so, in fact, $\hat{x}_\varepsilon(t_1)$ represents the minimax estimate of $Fx(t_1)$ for all ℓ verifying conditions of Theorem 2.4. (2.53) implies that the minimax error is given by:

$$\mathscr{I}^*(t_1) = \inf_u \sup_{f\in\mathscr{G},\eta} E(\ell_1^T(x_1(t_1) - \hat{x}_\varepsilon(t_1)))^2 \le \ell_1^T K_\varepsilon(t_1)\ell_1\,,$$

where we assumed that $F = \begin{bmatrix} I_r & 0_{r\times n-r} \\ 0_{m-r\times r} & 0_{m-r\times n-r}\end{bmatrix}$ and x_1 is a part of $x(t_1)$ corresponding to I_r. Thus, Riccati matrix $K_\varepsilon(t_1)$ defines the minimax error for $\ell_1^T x_1(t_1)$. If $\mathscr{I}^*(t_1) < +\infty$ for any ℓ_1 then $\|K_\varepsilon^{-\frac{1}{2}}(t_1)(x_1(t_1) - \hat{x}_\varepsilon(t_1))\| \le 1$ for any $f \in \mathscr{G}(t_1)$ and so the reachability set of DAE (1.3) is contained in the ellipsoid defined by $K_\varepsilon^{-\frac{1}{2}}(t_1)$ and centered around $\hat{x}_\varepsilon(t_1)$. Exact representation of the reachability set for singular stationary DAEs was constructed in [3]. Assumptions 1) and 2) (corollary 2.14) guarantee that the minimax error $\mathscr{I}^*(t_1)$ stays bounded for $t_1 \to \infty$. If the latter holds true for any ℓ_1 then the reachability set is contained in the ellipsoid defined by the solution of an algebraic Riccati equation. If one assumes $x(\cdot) \in \mathbb{H}_0^{1,\mathrm{F}}(t_0,+\infty)$ then $\ell_1^T x_1(t_1) \to 0$ for any ℓ_1 and so the trivial control $\hat{u} = 0$ provided by Corollary 2.13 is natural in this case.

[12] $\sup_\eta E\int_{t_0}^{t_1} u^T\eta dt = \int_{t_0}^{t_1} u^T R u dt$ by Cauchy inequality and so the first term in (1.4) "measures" the impact of the noise η onto $\sigma(t_1,\ell,u)$. If $R \to 0$ then the variation of the noise η goes to zero and the considered problem reduces to the classical observation problem in the control theory [31].

Let us consider a numerical example: assume F, C, H are defined by (2.22) and $R = I_3$, $Q = 0.02I_2$. Then ($\dot{x} := \frac{dx}{dt}$):

$$\begin{aligned} &\dot{x}_1 = x_3 + f_1, \quad \dot{x}_2 = -x_1 - x_4 + f_2, \quad x_1(t_0) = 0, \quad x_2(t_0) = 0\,, \\ &y_1 = x_1 + \eta_1, \quad y_2 = x_4 + \eta_2, \quad y_3 = x_2 + \eta_3\,. \end{aligned} \tag{2.54}$$

In order to generate $y_{1,2,3}$ we take $t_0 = 0$, $t_1 = 2$, $x_3 = \cos(t)$ and $x_4 = \sin(t)$, $f_{1,2}, \eta_{1,2,3} \sim \mathcal{U}(-2\sqrt{3}, 2\sqrt{3})$. Let us compute $\hat{u}$. According to Theorem 2.4 $\hat{u} = (\hat{u}_1, \hat{u}_2, \hat{u}_3)^T$ solves LQ-control problem (2.25)-(2.27), provided[13] $\ell_1 = 0$. Above (see example in subsection 2.3.1) we saw that $\hat{u}_{1,2} = -z_2$, and $\hat{u}_3 = p$, where p solves (2.29). Let k solve $\dot{k} = 150 - k^2$, $k(0) = 0$. We find that $\hat{u}_3 = kz_2$ where $\dot{z}_2 = kz_2$, $z_2(t_1) = \ell_2$. Let $\hat{x}$ be a solution to $\dot{\hat{x}} = -k\hat{x} - y_1 - y_2 + ky_3$, $\hat{x}(t_0) = 0$. Then it is easy to see that $\hat{u}(y) = \int_{t_0}^{t_1} \hat{u}^T y dt = \ell_2 \hat{x}(t_1)$. On the other hand, $\hat{u}_\varepsilon(y) = \ell_2 \hat{x}_{2,\varepsilon}(t_1)$, where $\hat{x}_\varepsilon(t_1) = (\hat{x}_{1,\varepsilon}(t_1), \hat{x}_{2,\varepsilon}(t_1))^T$ and $K_\varepsilon = \begin{bmatrix} k_{1,\varepsilon} & k_{2,\varepsilon} \\ k_{2,\varepsilon} & k_{4,\varepsilon} \end{bmatrix}$ solve (2.40) and (2.38) with $A_\varepsilon = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$, $Q_\varepsilon = (1+\varepsilon)I_2$, $S_\varepsilon = 50I_2$, $\Phi_\varepsilon = \begin{bmatrix} k_{1,\varepsilon} & k_{2,\varepsilon} & \frac{1}{\varepsilon} & 0 \\ k_{2,\varepsilon} & k_{4,\varepsilon} & 0 & \frac{-1}{1+\varepsilon} \end{bmatrix}$. We approximated $\hat{x}$ and $\hat{x}_\varepsilon$ integrating the corresponding ODEs numerically[14] over $[0, 2]$. We note that $K_\varepsilon(t)$ stabilizes around $t = 1.5$ so that the minimax error $\mathscr{I}^*(t)$ is bounded on infinity. The simulation results are presented in Figure 1.

3. Conclusion

This paper presents a generalization of Kalman duality principle for linear DAEs in the form (1.3). The only restriction we impose here is that F does not depend on time. Basically, our approach may be applied to the class of time-varying $F(t)$ with constant (or piece-wise constant) rank using Floke-Lyapunov theorem. The proof of the duality principle is based on basic notions of operator theory and Young-Fenchel duality (duality lemma 2.2). It reveals a connection between the minimax control problem (1.5) and dual control problem (2.10) for the adjoint DAE (2.9). It turns out that the problem (2.10) is ill-posed and classical optimality conditions are, therefore, not applicable in the general case. We present an example which shows that Pontryagin maximum principle selects a trivial control. The original problem has non-trivial solutions in $\mathbb{L}_2$, though. To overcome this we applied Tikhonov regularization approach in order to approximate a unique solution of the dual control problem. Tikhonov approach gave us necessary and sufficient optimality conditions for the LQ control problem with DAE constraints in the form of a two-point boundary value problem for a zero-index DAE. Using

[13] Although $y_1 = x_1 + \eta_1$ is observed, the minimax error $\mathscr{I}(t_1) = +\infty$ if $\ell_1 \neq 0$. This can be explained as follows. By (2.54) the derivative x_3 of x_1 may be any element of $\mathbb{L}_2$. As the expression for $\mathscr{I}^*(t_1)$ contains $\int_{t_0}^{t_1} x_3^T z_1 dt$ (see (2.16)) it follows that $\mathscr{I}^*(t_1) < +\infty$ if and only if $z_1(t) \equiv 0$ for any $t \in (t_0, t_1)$. As z_1 is absolutely continuous it follows that $z_1(t_1) = 0$. Hence, if $\ell_1 \neq 0$ then $\hat{u}_1 = \delta(t_1 - t)\ell_1$. $\delta \notin \mathbb{L}_2$ though. In this case, $\lim_{\varepsilon \downarrow 0} \|\hat{u}_\varepsilon\|_{\mathbb{L}_2} = +\infty$.

[14] Runge-Kutta method of 4th order with time-step $dt = 10^{-3}$

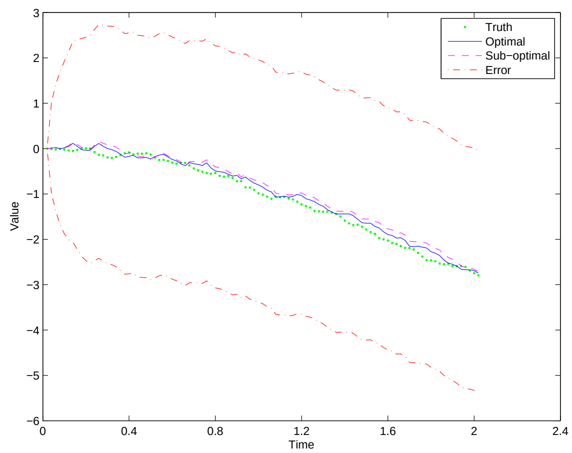

FIGURE 1. Optimal $\hat{u}(y) = \hat{x}(t)$ (solid), sub-optimal $\hat{u}_\varepsilon(y) = \hat{x}_{2,\varepsilon}(t)$ (dashed), "truth" $x_2(t)$ (dotted) and minimax error $\mathscr{I}^*(t)$ (DotDashed), $t \in [0, 2]$, $\varepsilon = 10^{-6}$. The sub-optimal control shows good scores: $\|\hat{x} - \hat{x}_{2,\varepsilon}\|_{L_2(0,2)} \approx 0.07$ and $\frac{\|\hat{x}(2) - \hat{x}_{2,\varepsilon}(2)\|}{\|\hat{x}(2)\|} \approx 0.01$.

this property we converted zero-index DAE into an equivalent ODE by means of singular value decomposition. Splitting the boundary value problem for an ODE we constructed Riccati equation on a subspace which corresponds to the differential part of DAE (1.3). As a result we obtained a minimizing sequence of controls $\{\hat{u}_\varepsilon\}$ in a feed-back form. This sequence converges in $\mathbb{L}_2$ if the adjoint DAE (2.9) is solvable for a given ℓ and converges weakly to a linear combination of distributions otherwise. In the case of infinite horizon problem we found that the cost function (1.4) is bounded from above by a maximal eigen value of the Riccati matrix. Also we presented $\hat{u}_\varepsilon$ in the dual form.

In perspective, it is important to define a generic procedure for choosing the regularization parameter ε depending on the time discretization step for DAE. General remarks on this topic are available in [29]. Another key point is to find an efficient description of a set $\mathscr{M}(t_1)$ of all ℓ such that the adjoint DAE with variable $C(t)$ has a solution. This problem was solved in [3] provided $C(t) \equiv C$.

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