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INVERSE PROBLEMS FOR LINEAR ILL-POSED DIFFERENTIAL-ALGEBRAIC EQUATIONS WITH UNCERTAIN PARAMETERS

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Abstract. This paper describes a minimax state estimation approach for linear differential-algebraic equations (DAEs) with uncertain parameters. The approach addresses continuous-time DAEs with non-stationary rectangular matrices and uncertain bounded deterministic input. An observation's noise is supposed to be random with zero mean and unknown bounded correlation function. Main result is a Generalized Kalman Duality (GKD) principle, describing a dual control problem. Main consequence of the GKD is an optimal minimax state estimation algorithm for DAEs with non-stationary rectangular matrices. An algorithm is illustrated by a numerical example for 2D time-varying DAE with a singular matrix pencil.

1. **Introduction.** In this paper we focus on an inverse problem for a linear Differential-Algebraic Equation (DAE) in the form

$$\frac{d(Fx)}{dt} = C(t)x(t) + f(t) \tag{1}$$

where F is a rectangular matrix. We note that the case $det(sF - C) \neq 0$ with $C(t) \equiv C$ is well-understood: by a non-singular linear transformation the matrix pencil $sF - C$ can be converted into a Kronecker canonical form. Accordingly, changing the basis in the state space of (1) and differentiating exactly d times (d is an index of the pencil $sF - C$), one can reduce (1) to an equivalent Ordinary Differential Equation (ODE), provided $f(\cdot)$ is sufficiently smooth and meets some algebraic constraints. The latter ODE has a unique solution, provided the initial condition belongs to some subspace. Details of the reduction process are presented in [7]. We will refer such DAEs as causal.

One reason to study non-causal DAEs comes from the state estimation theory. Namely, applying the optimal linear proportional feedback $f = Kx$ to (1) with a regular pencil ($det(sF - C) \neq 0$), one could arrive [19] to the system with a singular pencil $det(sF - C - BK) \equiv 0$. On the other hand, due to behavioral approach[10] one can treat an input f of the ODE $\frac{dx}{dt} = C(t)x(t) + f(t)$ as a part of the state so that the extended state (x, f) verifies DAE (1) with $F = [\,I \;\; 0\,]$ and $C(t) = [\,C(t) \;\; I\,]$. Using such a point of view one can convert parameter estimation problem

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for ODE into state estimation problem for non-causal DAE (see for instance [5] for the estimation of the unknown input). In addition, state estimation methods for ODE, based on the Kalman filtering approach, cannot be applied directly in the practical inverse problems due to the computational burden. For instance, state vector in meteorological applications usually has over 10^6 components, which makes it impossible to propagate a Riccati gain without appropriate reduction. A rigorous reduced-order state estimation (allowing to control the error, introduced by reduction) can be done by means of the state estimation framework for DAEs (main idea and its application to an operational air quality model could be found in [17]).

However, there are difficulties, connected with state estimation problems for DAEs. In particular, it is not clear when the seminal Kalman duality holds, that is the observation problem is equal to the dual control problem.

In addition, for the case of variable $C(t)$ there is a lack of optimality criteria for an LQ-control problems with DAE constraints. Sufficient optimality conditions in the form of the descriptor Euler-Lagrange equation were formulated in [13]. The latter can be solved numerically only for index-one causal DAEs. Existence of the optimal feedback solution was studied in [12]. The authors construct the feedback control by means of a Descriptor Riccati Equation. However, the authors point out that "numerically qualified versions of the Riccati DAE system" require further research.

The research, presented in this paper, resolves the above problems for a class of non-causal DAEs. In what follows, we apply a minimax estimation approach, developed in[21] for abstract linear equations in a Hilbert space, for non-causal DAE (1) with $F, C(t) \in \mathbb{R}^{m\times n}$. We will be interested in the following inverse problem: given observations $y(t)$, $t \in [t_0, T]$ of $x(\cdot)$, to reconstruct[1] $Fx(T)$, provided $x(\cdot)$ is a weak solution to (1), $f(\cdot)$ is some unknown squared-integrable function, which belongs to a given bounded set G. We will also assume that $y(t)$ is corrupted by a random noise $\eta(\cdot)$ with zero mean $E\eta(t) = 0$ and the correlation function $(t, s) \mapsto \mathcal{R}_\eta(t, s) := E\Psi(t)\Psi'(s)$ belongs to a given bounded set W.

Note that in order to reconstruct $Fx(T)$ it is enough[2] to reconstruct a linear function $\ell(x) := \langle \ell, Fx(T)\rangle$ for any $\ell \in \mathbb{R}^m$. In what follows, we will be looking for the estimate of $\ell(x)$ among all linear functions of observations $u(y)$ by means of the minimax state estimation approach[2, 16, 15, 4, 18]. The main idea is to find the estimate u with minimal worst-case error. Namely, for the fixed u the estimation error is $E[\ell(x) - u(y)]^2$. Since $E[\ell(x) - u(y)]^2$ depends on realizations of f and $\mathcal{R}_\eta$, it follows that, for a fixed u, the worst-case estimation error is the maximum of $E[\ell(x) - u(y)]^2$ over all possible realizations of f, $\mathcal{R}_\eta$ and[3] x: $\sigma(T, \ell, u) := \sup_{f\in G,\mathcal{R}_\eta\in W,x}\{E[\ell(x) - u(y)]^2\}$. Finally, $\hat{u}$ is called a minimax mean-squared estimate of $\ell(x)$, iff $\hat{u}$ has a minimal worst-case error (so called minimax error), that is $\sigma(T, \ell, \hat{u}) = \inf_u \sigma(T, \ell, u)$.

The minimax estimate for (1) was constructed in [18], provided $F = I$. The case of deterministic $\eta(\cdot)$ and $F = I$ was addressed in [2, 16, 15, 4], where the minimax estimation of $x(T)$ is shown to be the Tchebysheff center of the ODEs reachable set,

[1]Due to the definition given at the beginning of Section 2 just $Fx(\cdot)$ is supposed to be absolutely continuous. Hence $x(T)$ is not necessary defined at the time T. Thus we are searching for the estimate of $Fx(T)$.

[2]Having the estimate of $\ell(x)$ for any $\ell \in \mathbb{R}^m$, one can set $\ell := e_i = (0, \dots, 1, \dots 0)^T$ in order to reconstruct i-th component of $Fx(T)$.

[3]As DAE is non-causal, it could have more than one solution x, corresponding to the given f through (1).

consistent with observations and uncertainty description. The same result can be proved by the method of this paper for $F \neq I$, provided η is deterministic and (η, f) belongs to some ellipsoid. Main distinctive features of the problem, described above are: 1) as $F \in \mathbb{R}^{m\times n}$ it follows that DAE (2) cannot be converted into ODE in the general case; the latter makes it impossible to directly apply classical minimax state estimation methods [18, 2, 16, 15, 4] as it was done in [8, 5] for causal DAEs; 2) we consider a quite general noise η, which can be a realization of any random process Ψ (not necessary Gaussian as in [8]) with bounded correlation function.

The major contribution of this paper is a Generalized Kalman Duality (GKD) principle for non-causal time-dependent DAEs (Theorem 2.2). It states that the observation problem for DAEs is equal to some control problem with DAE constraints. The latter generalizes seminal Kalman duality for ODE and reflects the procedure of deriving the minimax estimate: to compute the minimax error $\inf_u \sigma(T, \ell, u)$ one needs to solve a control problem for the "adjoint" DAE. GKD was previously applied in [22] in order to construct the minimax estimate for non-causal DAEs with discrete time.

Following the GKD we derive a dual control problem for DAEs (Proposition 1) and we describe the minimum point as a solution of the Euler-Lagrange system, provided the matrices of DAE have "some regularity" (Proposition 1). Finally, an ODE, describing the minimax estimate (Corollary 1), is derived. The dimension of the state space of this ODE is equal to the rank of F, providing a reduced order estimate. In fact, the minimax estimate gives a reconstruction of the projection of $Fx(T)$ onto MOS $\mathcal{L}(T)$ only. As $\dim\mathcal{L}(T) \leq \operatorname{rank}F$, the minimax estimate introduces an additional degree of reduction. We finish with a numerical example.

Notation: $E\eta$ denotes the mean of the random element η; $\operatorname{int} G$ denotes the interior of G; $f(\cdot)$ or f denotes a function (as an element of some functional space); $f(t)$ denotes a value of function f at t; $\mathbb{R}^n$ denotes the n-dimensional Euclidean space over real field; $\mathbb{L}_2(t_0, T, \mathbb{R}^m)$ denotes a space of square-integrable functions with values in $\mathbb{R}^m$; $\mathbb{H}_2^1(t_0, T, \mathbb{R}^m)$ denotes a space of absolutely continuous functions with L_2-derivative and values in $\mathbb{R}^m$; the prime $'$ denotes the operation of taking the adjoint: L' denotes adjoint operator, F' denotes the transposed matrix; $c(G, \cdot)$ denotes the support function of some set G; $\langle \cdot, \cdot \rangle$ denotes the inner product in Hilbert space $\mathcal{H}$; $S > 0$ means $\langle Sx, x \rangle > 0$ for all x; F^+ denotes the pseudoinverse matrix, $\operatorname{tr}(S)$ denotes the trace of the matrix S.

2. Main result.

We proceed with a problem statement. Consider a pair of systems

$$\frac{d(Fx)}{dt} = C(t)x(t) + f(t), \quad Fx(t_0) = 0, \tag{2}$$

$$y(t) = H(t)x(t) + \eta(t), \tag{3}$$

where $x(t) \in \mathbb{R}^n$, $f(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$, $\eta(t) \in \mathbb{R}^p$ represent the state, input, observation and observation's noise respectively, $F \in \mathbb{R}^{m\times n}$, $f(\cdot) \in \mathbb{L}_2(t_0, T, \mathbb{R}^m)$, $C(t)$ and $H(t)$ are continuous functions of t on $[t_0, T]$, $t_0, T \in \mathbb{R}$.

$x(\cdot)$ is said to be a weak solution to DAE (2), if $Fx(\cdot)$ is an absolutely continuous function, $\dfrac{d}{dt}Fx$ is a squared-integrable vector-function on (t_0, T), $x(\cdot)$ verifies (2) a.e. on $[t_0, T]$ and[4] $Fx(t_0) = 0$.

[4] We have chosen the case with zero initial condition in order to simplify the presentation. The general case $Fx(t_0) = x_0$ can be treated by means of the same approach.

We assume $\eta(\cdot)$ in (3) is a realization of a random process Ψ such that $E\Psi(t) = 0$ on $[t_0, T]$ and

$$\mathcal{R}_\eta \in W = \{\mathcal{R}_\eta : \int_{t_0}^{T} \operatorname{tr}(R(t)\mathcal{R}_\eta(t,t))dt \le 1\} \tag{4}$$

The input $f(\cdot)$ in (2) is supposed to be deterministic and

$$f(\cdot) \in G := \{f(\cdot) : \int_{t_0}^{T} \langle Q(t)f(t), f(t)\rangle \le 1\}, \tag{5}$$

where $Q(t) \in \mathbb{R}^{m\times m}$, $Q = Q' > 0$, $R(t) \in \mathbb{R}^{p\times p}$, $R' = R > 0$; $Q(t)$, $R(t)$, $R^{-1}(t)$ and $Q^{-1}(t)$ are continuous functions of t on $[t_0, T]$.

Definition 2.1. Given $T < +\infty$, $u(\cdot) \in \mathbb{L}_2(t_0, T, \mathbb{R}^p)$ and $\ell \in \mathbb{R}^m$ define a mean-squared worst-case estimation error

$$\sigma(T, \ell, u) := \sup_{x(\cdot), f(\cdot)\in G, \mathcal{R}_\eta \in W} \{E[\langle \ell, Fx(T)\rangle - u(y)]^2\}$$

A function $\hat{u}(y) = \int_{t_0}^{T} \langle \hat{u}(t), y(t)\rangle dt$ is called an a priori minimax mean-squared estimate in the direction ℓ (ℓ-estimation) if $\inf_u \sigma(T, \ell, u) = \sigma(T, \ell, \hat{u})$. The number $\hat{\sigma}(T, \ell) = \inf_u \sigma(T, \ell, u)$ is called a minimax mean-squared a priori error in the direction ℓ at time-instant T (ℓ-error). The set $\mathcal{L}(T) = \{\ell \in \mathbb{R}^m : \hat{\sigma}(T, \ell) < +\infty\}$ is called a minimax observable subspace.

2.1. **Generalized Kalman Duality Principle.** Definition 2.1 generalizes the notion of the a priori minimax mean-squared estimation, introduced in [18]. Next theorem generalizes the celebrated Kalman duality principle [3] to non-causal DAEs.

Theorem 2.2 (Generalized Kalman Duality)**.** *The ℓ-error is finite iff*

$$\frac{d(F'z)}{dt} = -C'(t)z(t) + H'(t)u(t), \quad F'z(T) = F'\ell \tag{6}$$

for some $z(\cdot)$ and $u(\cdot)$. In this case the problem $\sigma(T, \ell, u) \to \inf_u$ is equal to the following optimal control problem

$$\sigma(T, \ell, u) = \min_v \{\int_{t_0}^{T} \langle Q^{-1}(z - v), z - v\rangle dt\} + \int_{t_0}^{T} \langle R^{-1}u, u\rangle dt \to \min_u, \tag{7}$$

with constraint (6), provided $v(\cdot)$ obeys (6) with $u(\cdot) = 0$ and $\ell = 0$.

Remark 1. An obvious corollary of the Theorem 2.2 is an expression for the minimax observable subspace

$$\mathcal{L}(T) = \{\ell \in \mathbb{R}^n : F'z(T) = F'\ell, \frac{d(F'z)}{dt} + C'z - H'u = 0 \text{ for some } z(\cdot), u(\cdot)\}$$

In the case of stationary $C(t)$ and $H(t)$ the minimax observable subspace may be calculated explicitly, using the canonical Kronecker form [7].

Proof of Theorem 2.2. Take $\ell \in \mathbb{R}^m$, $u(\cdot) \in \mathbb{L}_2(t_0, T, \mathbb{R}^p)$ and suppose ℓ-error is finite. The proof is based on a generalized integration-by-parts formula

$$\begin{aligned}\langle F'w(T), F^+Fx(T)\rangle - \langle F'w(t_0), F^+Fx(t_0)\rangle \\ = \int_{t_0}^{T} \langle \frac{d(Fx)}{dt}, w\rangle + \langle \frac{d(F'w)}{dt}, x\rangle dt\end{aligned} \tag{8}$$

proved in [20] for $Fx(\cdot) \in \mathbb{H}_2^1(t_0,T,\mathbb{R}^m)$ and $F'w(\cdot) \in \mathbb{H}_2^1(t_0,T,\mathbb{R}^n)$. There exists[5] $w(\cdot) \in \mathbb{L}_2(t_0,T,\mathbb{R}^m)$ such that $F'w(\cdot) \in \mathbb{H}_2^1(t_0,T,\mathbb{R}^n)$ and $F'w(T) = F'\ell$. Noting that [1] $F = FF^+F$ and combining (8) with (2) one derives

$$\begin{aligned}\langle \ell, Fx(T)\rangle &= \langle F'\ell, F^+Fx(T)\rangle = \langle F'w(T), F^+Fx(T)\rangle \\ &= \int_{t_0}^T \langle \frac{d(Fx)}{dt}, w\rangle + \langle \frac{d(F'w)}{dt}, x\rangle dt = \int_{t_0}^T \langle f, w\rangle + \langle \frac{d(F'w)}{dt} + C'w, x\rangle dt\end{aligned} \tag{9}$$

Combining (9) with $E\eta(t) = 0$ we have

$$\begin{aligned}E[\langle \ell(x) - u(y)]^2 &= [\langle \ell, Fx(T)\rangle - \int_{t_0}^T \langle H'u, x\rangle dt]^2 + E[\int_{t_0}^T \langle u(t), \eta(t)\rangle dt]^2 \\ &= [\int_{t_0}^T \langle f, w\rangle + \langle \frac{d(F'w)}{dt} + C'w - H'u, x\rangle dt]^2 + E[\int_{t_0}^T \langle u(t), \eta(t)\rangle dt]^2\end{aligned} \tag{10}$$

Using Cauchy inequality one derives $E(\int_{t_0}^T \langle u, \eta\rangle dt)^2 \le \int_{t_0}^T E\langle R\eta, \eta\rangle dt \int_{t_0}^T \langle R^{-1}u, u\rangle dt$. As $E\langle R\eta, \eta\rangle = \mathrm{tr}\,(R(t)\mathcal{R}_\eta(t,t))$, it follows from (4)

$$E(\int_{t_0}^T \langle u, \eta\rangle dt)^2 \le \int_{t_0}^T E\langle R\eta, \eta\rangle dt \int_{t_0}^T \langle R^{-1}u, u\rangle dt \le \int_{t_0}^T \langle R^{-1}u, u\rangle dt$$

so that

$$\sup_{\mathcal{R}_\eta \in W} E[\int_{t_0}^T \langle u(t), \eta(t)\rangle dt]^2 = \int_{t_0}^T \langle R^{-1}u, u\rangle dt$$

This and $\hat\sigma(T,\ell) < +\infty$ imply the first term in the last line of (10) is bounded. $\int_{t_0}^T \langle f, w\rangle dt$ does not depend on $x(\cdot)$ and is bounded due to (5). Therefore

$$\sup_{x(\cdot)} \{\int_{t_0}^T \langle \frac{d(F'w)}{dt} + C'w - H'u, x\rangle dt : \frac{d(Fx)}{dt} = Cx + f, f(\cdot) \in G\} < +\infty \tag{11}$$

In fact, this observation allows us to prove that there exists $z(\cdot)$ such that (6) holds for the given ℓ and $u(\cdot)$. To do so we apply a general duality result[6] of [21]:

$$\sup_{x \in \mathscr{D}(L)} \{\langle \mathcal{F}, x\rangle, Lx \in G\} = \inf_{b \in \mathscr{D}(L')} \{c(G, b), L'b = \mathcal{F}\} \tag{12}$$

provided (A1) $L : \mathscr{D}(L) \subset \mathcal{H}_1 \to \mathcal{H}_2$ is a closed dense defined linear mapping, (A2) $G \subset H_2$ is a closed bounded convex set and $\mathcal{H}_{1,2}$ are abstract Hilbert spaces.

Define

$$Lx = \frac{d(Fx)}{dt} - Cx,\ x \in \mathscr{D}(L) := \{x(\cdot) : Fx \in \mathbb{H}_2^1(t_0,T,\mathbb{R}^n), Fx(t_0) = 0\} \tag{13}$$

It was proved in [20], that L is closed dense defined linear mapping and

$$L'b = -\frac{d(F'b)}{dt} - C'b,\ b \in \mathscr{D}(L') := \{b(\cdot) : F'b \in \mathbb{H}_2^1(t_0,T,\mathbb{R}^m), F'b(T) = 0\} \tag{14}$$

Setting $\mathcal{F} := \dfrac{d(F'w)}{dt} + C'w - H'u$ we see from (11) that the right-hand part of (12) is finite. Using (14) one derives

$$\inf\{c(G,b), -\frac{d(F'b)}{dt} - C'(t)b(t) = \frac{d(F'w)}{dt} + C'w - H'u\} < +\infty \tag{15}$$

[5] A trivial example is $w(t) \equiv \ell$.

[6] (12) was proved in [11] for bounded L and Banach space $\mathcal{H}_{1,2}$

(15) implies $\frac{d(F'z)}{dt}+C'z = H'u$, $F'z = F'\ell$ with $z := (w+b)$ and some $b(\cdot) \in \mathscr{D}(L')$, verifying the equality in (15). This proves (6) has a solution $z(\cdot)$.

On the contrary, let $z(\cdot)$ verify (6) for the given ℓ and $u(\cdot)$. Using (10) one derives

$$\sigma(T,\ell,u) = \big[\sup_{f\in G_1} \int_{t_0}^{T} \langle z, f\rangle dt\big]^2 + \int_{t_0}^{T} \langle R^{-1}u, u\rangle dt < \infty \tag{16}$$

provided $G_1 = G\cap\mathcal{R}(L)$, $\mathcal{R}(L)$ is the range of the linear mapping L defined by (13) (G_1 is a set of all $f(\cdot)$ such that $f(\cdot)$ verifies (5) and (2) has a solution $x(\cdot)$).

To prove the rest of the theorem we will apply another duality result of [21, 11]:

$$\sup\{\langle f, z\rangle, f \in G \cap \mathcal{R}(L)\} = \inf\{c(G, z - v), v \in N(L')\} \tag{17}$$

provided L, G verify (A1), (A2) and $\operatorname{int} G \cap \mathcal{R}(L) \neq \varnothing$. It is easy to see that the latter inclusion holds for L and G defined by (13) and (5) respectively. Recalling the definition of L' (formula (14)) and noting that

$$c^2(G, z - v) = \int_{t_0}^{T} \langle Q^{-1}(z - v), z - v\rangle dt$$

we derive from (16)-(17) that $\sigma(T,\ell,u) \to \min_u$ is equal to (7). □

3. **Optimality conditions and estimation algorithm.** Due to Theorem 2.2 the minimax state estimation problem is equal to the dual control problem (7) with DAE constrain (6), provided $\ell \in \mathcal{L}(T)$. This result holds for any $F \in \mathbb{R}^{m\times n}$ and continuous $t \mapsto C(t) \in \mathbb{R}^{m\times n}$. Therefore, in order to find the ℓ-estimation $\hat{u}$ we need to solve (7). In what follows, we formulate optimality conditions for (7) in the form of a boundary value problem (BVP) for a descriptor Euler-Lagrange equation. We present a condition on the matrices $C(t)$, $H(t)$ and F, which allow one to convert the latter BVP for DAE into an equivalent BVP for ODE. Our approach is a modification of a splitting method, discussed in [6]. We will split DAE into differential and algebraic parts applying SVD decomposition [1] to F. Let $D = \operatorname{diag}(\lambda_1 \dots \lambda_r)$ where λ_i are positive eigen values of FF', $i \in \{1, \dots, r := \operatorname{rang} F\}$. Then there exist $S_L \in \mathbb{R}^{m\times m}$, $S_R \in \mathbb{R}^{n\times n}$ such that

$$F = S_L\Lambda S_R, S_L S_L' = I, S_R S_R' = I, \Lambda = \begin{bmatrix} D^{\frac{1}{2}} & 0_{r\times n-r} \\ 0_{m-r\times r} & 0_{m-r\times n-r}\end{bmatrix} \tag{18}$$

Transforming (2) according to (18) and changing the variables one can reduce the general case to the case $F = \left[\begin{smallmatrix} I & 0 \\ 0 & 0\end{smallmatrix}\right]$. Therefore, without loss of generality, we will assume $F = \left[\begin{smallmatrix} I & 0 \\ 0 & 0\end{smallmatrix}\right]$. We split ℓ, $C(t)$, $Q(t)$ and $H'(t)R(t)H(t)$ accordingly: $C(t) = \left[\begin{smallmatrix} C_1 & C_2 \\ C_3 & C_4\end{smallmatrix}\right]$, $Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2' & Q_4\end{bmatrix}$, $H'RH = \begin{bmatrix} S_1 & S_2 \\ S_2' & S_4\end{bmatrix}$, $\ell = (\ell_1, \ell_2)$. Define $W(t) = (S_4 + C_4'Q_4^{-1}C_4)$, $A(t) = (C_3'Q_4^{-1}C_4 + S_2)$ and $B(t) = (C_2' - C_4'Q_4^{-1}Q_2')$.

Proposition 1 (optimality conditions)**.** *Let $\mathcal{R}(C_2'(t)) \subseteq \mathcal{R}(C_4'(t))$ and*

$$\int_{t_0}^{T} \|W^{+}(t)B(t)g(t)\|^2 dt < \infty, \quad \forall g(\cdot) \in \mathbb{L}_2(t_0, T, \mathbb{R}^r)$$

Then the Euler-Lagrange system

$$\begin{aligned} \frac{d(Fp)}{dt} &= C(t)p(t) + Q^{-1}(t)z(t), \quad Fp(t_0) = 0, \\ \frac{d(F'z)}{dt} &= -C'(t)z(t) + H'(t)R(t)H(t)p(t), \quad F'z(T) = F'\ell \end{aligned} \tag{19}$$

has a solution for any $\ell \in \mathbb{R}^n$*. If* $p(\cdot)$ *and* $z(\cdot)$ *are some solution of* (19) *then, the* ℓ*-estimation is given by* $\hat{u} = RHp$ *and the* ℓ*-error* $\hat{\sigma}(T, \ell) = \langle F'^{+}\ell, Fp(T)\rangle$*.*

For instance, condition on W and B of the theorem is satisfied, if $C(t)$ meets the conditions of [9].

Proof. Taking into account the above splitting we rewrite (19) in the following form

$$\begin{aligned} &\frac{dp_1}{dt} = C_1p_1 + Q_1z_1 + C_2p_2 + Q_2z_2, \quad p_1(t_0) = 0, \\ &\frac{dz_1}{dt} = -C_1'z_1 + S_1p_1 - C_3'z_2 + S_2p_2, \quad z_1(T) = \ell_1, \\ &0 = C_3p_1 + C_4p_2 + Q_2'z_1 + Q_4z_2, \\ &0 = -C_2'z_1 - C_4'z_2 + S_2'p_1 + S_4p_2. \end{aligned} \tag{20}$$

As $Q^{-1} > 0$, it follows that $Q_4 > 0$. Hence $z_2 = -Q_4^{-1}(C_3p_1 + C_4p_2 + Q_2'z_1)$ and

$$W(t)p_2 = B(t)z_1 - A'(t)p_1 \tag{21}$$

where A, B, W were defined above. Since $\mathcal{R}(C_2') \subset \mathcal{R}(C_4')$ it follows that (21) is point-vise solvable (in the algebraic sense) and one solution has the form $p_2 = W^{+}(t,0)(B(t)z_1 - A'(t)p_1)$ The second assumption imply $p_2 \in \mathbb{L}_2(t_0, T, \mathbb{R}^{n-r})$. Substituting p_2 into (20) and noting that $C_4(I - W^{+}(t)W(t)) = 0$ we obtain

$$\begin{aligned} &\frac{dp_1}{dt} = C_{+}(t)p_1 + S_{+}(t)z_1, \quad p_1(t_0) = 0 \\ &\frac{dz_1}{dt} = -C_{+}'(t)z_1 + Q_{+}(t)p_1, \quad z_1(T) = \ell_1 \end{aligned} \tag{22}$$

with $C_{+}(t) := C_1 - Q_2Q_4^{-1}C_3 - B'W^{+}(t)A'$, $S_{+}(t) := Q_1 - Q_2Q_1^{-1}Q_3 + B'W^{+}(t)B$, $Q_{+}(t) := S_1 + C_3'Q_4^{-1}C_3 - AW^{+}(t)A'$. Applying simple matrix manipulations one can prove that $S_{+} \geq 0$, $Q_{+} \geq 0$ so that (22) is a non-negative Euler-Lagrange system in the Hamilton's form. Therefore it is always solvable [14]. We continue with the second part of the proposition.

Due to GKD (Theorem 2.2), it is sufficient to show that $\hat{u}$, defined in the statement of Proposition, solves (7). We note that (7) is equal to

$$N(z,u) := \int_{t_0}^{T} \langle Q^{-1}z, z\rangle dt + \int_{t_0}^{T} \langle R^{-1}u, u\rangle dt \to \min_{(z,u) \text{ verify } (6)} \tag{23}$$

Therefore, to conclude it is sufficient to show that $N(z,u) - N(\hat{u}, \hat{z}) \geq 0$, where $\hat{u} = RHp$, $(\hat{p}, \hat{z})$ denote some solution of (19) and (z,u) verifies (6). We have

$$\begin{aligned} &\frac{1}{2}(N(z,u) - N(\hat{u},\hat{z})) = (\langle F'\hat{z}(T), F^{+}F\hat{p}(T)\rangle - \frac{1}{2}N(\hat{z},\hat{u})) \\ &- (\langle F'z(T), F^{+}F\hat{p}(T)\rangle - \frac{1}{2}N(z,u)) = \int_{t_0}^{T} \langle \frac{d(F'\hat{z})}{dt}, \hat{p}\rangle + \langle \frac{d(F\hat{p})}{dt}, \hat{z}\rangle dt - \frac{1}{2}N(\hat{z},\hat{u}) \\ &- (\int_{t_0}^{T} \langle \frac{d(F'z)}{dt}, \hat{p}\rangle + \langle \frac{d(F\hat{p})}{dt}, z\rangle dt - \frac{1}{2}N(z,u)) = \int_{t_0}^{T} \langle H'\hat{u}, \hat{p}\rangle + \|\hat{z}\|^2 dt - \frac{1}{2}N(\hat{z},\hat{u}) \\ &- (\int_{t_0}^{T} \langle H'u, \hat{p}\rangle + \langle z, \hat{z}\rangle dt - \frac{1}{2}N(z,u)) = \frac{1}{2}\int_{t_0}^{T} \|u - \hat{u}\|^2 + \|z - \hat{z}\|^2 dt \geq 0 \end{aligned}$$

□

Corollary 1 (sequential ℓ-estimation)**.** *Define C_+, Q_+ and S_+ as in (22) and let*

$$\begin{aligned}\frac{dK}{dt} &= C_+(t)K + KC_+(t)' - KQ_+(t)K + S_+(t), \quad K(t_0) = 0,\\ \frac{d\hat{x}}{dt} &= (C_+(t) - KQ_+(t))\hat{x} + \begin{bmatrix} K & (B'-KA)W^+(t)\end{bmatrix} HRy(t), \quad \hat{x}(t_0) = 0\end{aligned} \tag{24}$$

If $\hat{u}$ is defined as in Proposition 1 and $\ell = (\ell_1, \ell_2)$ (as in (22)) then

$$\int_{t_0}^{T} \langle \hat{u}, y\rangle = \langle \ell_1, \hat{x}(T)\rangle, \quad \hat{\sigma}(T,\ell) = \langle K(T)\ell_1, \ell_1\rangle \tag{25}$$

Proof. The proof uses the standard reduction procedure, that is a reduction of the Euler-Lagrange system (22) for (p_1, z_1) to some Cauchy problem for z_1, introducing a Riccati matrix K.

Assume z_1 solves $\frac{dz_1}{dt} = -C_+'(t)z_1 + Q_+(t)Kz_1$, $z_1(T) = \ell_1$. Define $p_1 := Kz_1$. By direct calculation one finds that (p_1, z_1) verify (22). As (22) is always uniquely solvable, it follows that the Riccati equation in (24) has a unique solution.

Let p_2 solves (21). Let us find z_2 from the third equation of (20) and set $\hat{p} = (p_1, p_2)^T$, $\hat{z} = (z_1, z_2)^T$. Then $(\hat{z}, \hat{p})$ solves (19) with $F = \left[\begin{smallmatrix} I & 0\\ 0 & 0\end{smallmatrix}\right]$. Proposition 1 implies that the minimax estimation is given by $\hat{u} = RH\hat{p}$ and the minimax error is $\hat{\sigma}(T,\ell) = \langle F'^{+}\ell, Fp(T)\rangle$. Now, integrating by parts $\int_{t_0}^{T}\langle \hat{u}, y\rangle$ and taking into account definitions of p_1, p_2, z_1 and $\hat{x}$ we arrive to the first equality in (25). Again, using integration by parts in $\langle F'^{+}\ell, Fp(T)\rangle$ one obtains the second equality in (25). □

3.1. Numerical example: non-causal non-stationary DAE.

Let

$$F = \left[\begin{smallmatrix} 1 & 0\\ 0 & 0\end{smallmatrix}\right], C(t) = \begin{bmatrix} -1 & 1\\ c_3(t) & 0\end{bmatrix}, H(t) = \begin{bmatrix} 0 & 1\end{bmatrix}.$$

Then $\det(F - \lambda C(t)) \equiv 0$ if $c_3(t) = 0$. Note that the pencil $sF' - C' - H'H$ is regular. The corresponding DAE reads

$$\begin{aligned}\frac{dx_1}{dt} &= -x_1 + x_2 + f_1(t),\\ 0 &= c_3(t)x_1(t) + f_2(t), \quad x_1(0) = 0\end{aligned} \tag{26}$$

Set $f_1 = 0$ for simplicity. We have $x_1(t) = \int_0^t \exp(s-t)x_2(s)ds$ and $c_3(t)\int_0^t \exp(s-t)x_2(s)ds = -f_2(t)$. From the latter formulae we see that the DAE (26) is ill-posed: x_1 is non-unique and is not continuous with respect to the input data[7].

Let us estimate $x_1(t)$, provided $y(t) = x_2(t) + \eta(t)$ is measured, (x_1, x_2) obeys (26) and (f_1, f_2, η) verify

$$\int_0^T f_1^2 + \frac{\exp(\sqrt{t})}{2} f_2^2 dt \le 1, \quad \mathrm{E}\int_0^T \frac{6}{T}\eta^2(t)dt \le 1, \quad \mathrm{E}\eta(\cdot) = 0 \tag{27}$$

As (19) is solvable, we can apply Corollary 1. The minimax error is given by $\hat{\sigma}(t,\ell) := K(t)\ell_1^2$ and minimax estimate has the following form

$$\begin{aligned}\frac{d\hat{x}}{dt} &= \big[-1 - K(t)\frac{2c_3^2(t)}{\exp(\sqrt{t})})\big]\hat{x} + y(t), \quad \hat{x}(0) = 0,\\ \frac{dK}{dt} &= -2K - \frac{2c_3^2(t)}{\exp(\sqrt{t})}K^2 + (1 + \frac{T}{6}), \quad K(0) = 0,\end{aligned}$$

[7] Since x_2 depends on derivative of f_2, it follows that x_2 is not $\mathbb{L}_2$-continuous with respect to f_2, implying ill-posedness. As x_2 depends on an arbitrary function v from some linear subspace we have non-uniqueness.

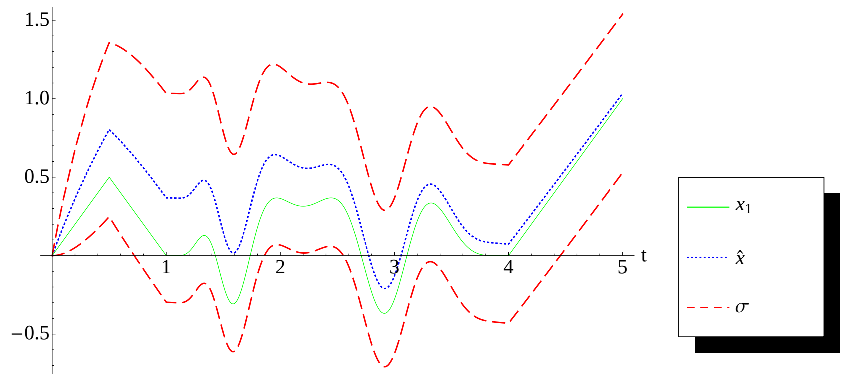

FIGURE 1. Minimax estimate $\hat{x}(t)$, error $\sigma(\ell_1, t)$ and the simulated $x_1(t)$ for $t \in [0, 5]$.

Figure 1 reflects the result of numerical simulations with particular f_2 and η, verifying (27).

4. **Conclusion.** The paper describes the minimax state estimation approach for linear non-causal DAEs, that is to find an estimate $u(y)$ of the linear function $\langle \ell, Fx(T)\rangle$, minimizing the worst-case error. The case of unknown but bounded input f and random observation error η with uncertain bounded correlation function is considered. The background of the approach is a Generalized Kalman Duality (GKD) principle. The GKD is used to calculate minimal worst-case (minimax) error. In contrast to causal DAEs, the minimax error could be infinite for some directions ℓ if the DAE is non-causal. In this case the observations $y(t)$ along with state equation (1) do not provide sufficient information for reconstructing $Fx(T)$. In fact, only a projection of $Fx(T)$ onto some subspace $\mathcal{L}(T)$, so called Minimax Observable Subspace (MOS), can be reconstructed. $\mathcal{L}(T)$ describes an "observable" (in the minimax sense) part of $Fx(T)$: MOS consists of all ℓ, for which minimax error is finite. As a consequence, for any linear estimate $u(y)$ of $\langle \ell, Fx(T)\rangle$ the estimation error varies[8] in $[0, +\infty]$ if $\ell \notin \mathcal{L}(T)$. For the case of constant $C(t)$ the MOS can be efficiently calculated (Remark 1).

Restricting the matrices of DAE we present sufficient solvability conditions for the Euler-Lagrange system, describing points of minimum of the dual control problem. This, in turn, allows to derive a reduced-order minimax estimate in the form of the minimax filter. The results are illustrated by a synthetic example of non-causal ill-posed 2D DAE.

[8]So that, for any natural N there is a realization of uncertain parameters $f(\cdot)$ and $\eta(\cdot)$ such that the estimation error will be greater than N

It would be interesting to derive a sub-optimal minimax estimate for DAEs without restrictions of Proposition 1. In particular, such estimate can be useful for the generation of the robust and mathematically justified reduced order state estimate for systems with a high dimension of the state vector. The latter can be done by projecting the state of the system onto some subspace (defined, for instance, by Proper Orthogonal decomposition) and to apply minimax estimation algorithm for a resulting DAE [17].

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