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Infinite horizon control and minimax observer design for linear DAEs

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Abstract— **In this paper we construct an infinite horizon minimax state observer for a linear stationary differential-algebraic equation (DAE) with uncertain but bounded input and noisy output. We do not assume regularity or existence of a (unique) solution for any initial state of the DAE. Our approach is based on a generalization of Kalman's duality principle. In addition, we obtain a solution of infinite-horizon linear quadratic optimal control problem for DAE.**

I. Introduction

Consider a linear Differential-Algebraic Equation (DAE) with state x, output y and noises f and η:

$$\frac{d(Fx)}{dt} = Ax(t) + f(t), \quad Fx(t_0) = x_0\,,$$
$$y(t) = Hx(t) + \eta(t)$$

where $F, A \in \mathbb{R}^{n\times n}$, $H \in \mathbb{R}^{p\times n}$. We do not restrict DAE's coefficients, in particular, we do not require that it has a solution for any initial condition x_0 or that this solution is unique. The only assumption we impose is that x_0, f and η are uncertain but bounded and belong to an ellipsoid. We would like to estimate a component $\ell^T Fx(t)$, $\ell \in \mathbb{R}^n$ of the DAE's state based on the output y. The desired *observer* should be linear in y, i.e. we are looking for maps $U(t,\cdot) \in L^2$ such that the estimate of $\ell^T Fx(t)$ at time t is of the form $\int_0^t U(t,s)y(s)ds$. The goal of the paper is to find an observer U such that:

1) The worst-case asymptotic observation error $\limsup_{t\to\infty} \sup_{f,\eta}(\ell^T Fx(t) - \int_0^t U(t,s)y(s)ds)^2$ is minimal, and
2) U can be implemented by a stable LTI system, i.e. the estimate $t \mapsto \int_0^t U(t,s)y(s)ds$ should be the output of a stable LTI system whose input is y.

We will call the observers defined above *minimax observers*.

Motivation The minimax approach is one of many classical ways to pose a state estimation problem. We refer the reader to [12], [4], [14] and [9] for the basic information on the minimax framework. Apart from pure theoretical reasons our interest in the minimax problem is motivated by applications of DAE state estimators in practice. In [22] we briefly discussed one application of DAEs to non-linear filtering problems. Namely, it is well known (see [6]) that the density of a wide class of non-linear diffusion processes solves forward Kolmogorov equation. The latter is a linear parabolic PDE and its analytical solution is usually unavailable. Different approximation techniques exist, though. One can project the density onto a finite dimensional subspace and derive a DAE for the projection coefficients. The resulting DAE will contain additive noise terms which represent the projection error (see [11], [20] for details). The minimax observer for this DAE can be used to construct a state estimate for the non-linear diffusion process.

Besides, DAEs have a wide range of applications, without claiming completeness, we mention robotics [16], cyber-security [15] and modeling various systems [13].

Contribution of the paper In this paper we follow the procedure proposed in [22]: we apply a generalization of Kalman's duality principle in order to transform the minimax estimation problem into a dual optimal control problem for the adjoint DAE. The latter control problem is an infinite horizon linear quadratic optimal control problem for DAEs. Duality allows us to view the observer U as a control input for the adjoint system and consider the worst-case estimation error $\limsup_{t\to\infty} \sup_{f,\eta}(\ell^T Fx(t) - \mathcal{O}_U(t))^2$ as the quadratic cost function of the dual control problem. Thus, the solution of the dual problem yields an observer whose worst-case asymptotic error is the minimal one. The resulting control problem is solved by translating it to a classical optimal control problem for LTIs. The feed-back solution of the latter yields a stable autonomous LTI systems, whose output is the minimax observer. The translation of the dual control problem to an LTI control problem relies on linear geometric control theory [17], [2]: the state and input trajectories of the DAE correspond to trajectories of an LTI restricted to its largest output zeroing subspace. To sum up, in this paper we solve **(1)** minimax estimation problem, and **(2)** infinite horizon optimal control problem for DAEs without restrictions on F and A.

Related work To the best of our knowledge, the results of this paper are new. The literature on DAE is vast, but most of the papers concentrate on regular DAEs. The papers [18], [5] are probably the closest to the current paper. However, unlike [18], we allow non-regular DAEs, and unlike [5], we do not require impulsive observability. In addition, the solution methods are also very different. The finite horizon minimax estimation problem and the corresponding optimal control problem for general DAEs was presented in [22]. A different way of representing solutions of DAEs as outputs of a LTI were presented in [22] too. We note that a feed-back control for finite and infinite-horizon LQ control problems with stationary DAE constraints was constructed in [3] assuming that the matrix pencil $F - \lambda A$ was regular. It was mentioned in [22] that transformation of DAE into Weierstrass canonical form may require taking derivative of the model error f, which, in turn, leads to restriction of the admissible class of model errors. In contrast, our approach is valid for L^2-model errors, which makes it more attractive for applications. Generalized Kalman duality principle for non-stationary DAEs with non-ellipsoidal uncertainty description was introduced in [21] where it was applied to get a sub-optimal infinite-horizon observer. The infinite-horizon LQ control problem for non-regular DAE was also addressed in [19], but unlike this paper, there it is assumed that the DAE has a solution from any initial state. Optimal control of non-linear and time-varying DAEs was also addressed in the literature. Without claiming completeness we mention [8], [7].

Outline of the paper This paper is organized as follows. Subsection I-A contains notations, section II describes the mathematical problem statement, section III presents the main results of the paper.

A. Notation

$S > 0$ means $x^T Sx > 0$ for all $x \in \mathbb{R}^n$; F^+ denotes the pseudoinverse matrix. Let I be either a finite interval $[0, t]$ or the infinite time axis $I = [0, +\infty)$. We will denote by $L^2(I, \mathbb{R}^n)$, $L^2_{loc}(I, \mathbb{R}^n)$ the sets of all square-integrable, and locally square integrable functions $f : I \to \mathbb{R}^n$ respectively. Recall that a function is locally square integrable, if its restriction to any compact interval is square integrable. If I is a compact interval, then $L^2_{loc}(I, \mathbb{R}^n) = L^2(I, \mathbb{R}^n)$. If $\mathbb{R}^n$ is clear from the context and $I = [0, t]$, $t > 0$, we will use the notation $L^2(0, t)$ and $L^2_{loc}(0, t)$ respectively. If f is a function, and A is a subset of its domain, we denote by $f|_A$ the restriction of f to A. We denote by I_n the $n \times n$ identity matrix.

II. Problem statement

Assume that $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}^p$ represent the state vector and output of the following DAE:

$$\begin{aligned} &\frac{d(Fx)}{dt} = Ax(t) + f(t)\,, \quad Fx(0) = x_0\,, \\ &y(t) = Hx(t) + \eta(t)\,, \end{aligned} \tag{1}$$

where $F, A \in \mathbb{R}^{n\times n}$, $H \in \mathbb{R}^{p\times n}$, and $f(t) \in \mathbb{R}^n$, $\eta(t) \in \mathbb{R}^p$ stand for the model error and output noise respectively. In this paper we consider the following functional class for DAE's solutions: if x is a solution on some finite interval $I = [0, t_1]$ or infinite interval $I = (0, +\infty)$, then $x \in L^2_{loc}(I)$, and Fx is absolutely continuous. This allows to consider a state vector $x(t)$ with a non-differentiable part belonging to the null-space of F. We refer the reader to [21] for further discussion.

In what follows we assume that for any initial condition x_0 and any time interval $I = [0, t_1]$, $t_1 < +\infty$, model error f and output noise η are unknown and belong to the given ellipsoidal bounding set $\mathscr{E}(t_1) := \{(x_0, f, \eta) \in \mathbb{R}^n \times L^2(I, \mathbb{R}^n) \times L^2(I, \mathbb{R}^p) : \rho(x_0, f, \eta, t_1) \le 1\}$, where

$$\rho(x_0, f, \eta, t_1) := x_0^T Q_0 x_0 + \int_0^{t_1} f^T Q f + \eta^T R\eta dt\,, \tag{2}$$

and $Q_0, Q(t) \in \mathbb{R}^{n\times n}$, $Q_0 = Q_0^T > 0$, $Q = Q^T > 0$, $R \in \mathbb{R}^{p\times p}$, $R^T = R > 0$. In other words, we assume that the triple (x_0, f, η) belongs to the unit ball defined by the norm ρ.

First, we study the state estimation problem for finite time interval $[0, t_1]$. Our aim is to construct the estimate of the linear function of the state vector $\ell^T Fx(t_1)$, $\ell \in \mathbb{R}^n$, given the output $y(t)$ of (1), $t \in [0, t_1]$. Following [1] we will be looking for an estimate in the class of linear functionals

$$\mathcal{O}_{U,t_1}(y) = \int_0^{t_1} y^T(s)U(s)dt\,,$$

$U \in L^2(0, t_1)$. Such linear functionals represent linear estimates of a state component $\ell^T Fx(t_1)$ based on past outputs y. We will call functions $U \in L^2(0, t_1)$ *finite horizon observers*. With each observer U we will associate an observation error defined as follows:

$$\sigma(U, t_1, \ell) := \sup_{(x_0,f,\eta)\in\mathscr{E}(t_1)} (\ell^T Fx(t_1) - \mathcal{O}_{U,t_1}(y))^2\,.$$

The observation error $\sigma(U, \ell, t_1)$ represents the biggest estimation error of $\ell^T Fx(t_1)$ which can be produced by the observer U, if we assume that the initial state and the noise belong to $\mathscr{E}(t_1)$.

So far, we have defined observers which act on finite time intervals. Next, we will define an analogous concept for the whole time axis $[0, +\infty)$.

Definition 1 (Infinite horizon observers): Denote by $\mathscr{F}$ the set of all maps $U : \{(t_1, s) \mid t_1 > 0, s \in [0, t_1]\} \to \mathbb{R}^p$ such that for every $t_1 > 0$, the map $U(t_1, \cdot) : [0, t_1] \ni s \mapsto U(t_1, s)$ belongs to $L^2(0, t_1)$.

An element $U \in \mathscr{F}$ will be called an *infinite horizon observer*. If $y \in L^2_{loc}(I, \mathbb{R}^p)$, $I = [0, t_1]$, $t_1 > 0$ or $I = [0, +\infty)$, then the result of applying U to y is a function $O_U(y) : I \to \mathbb{R}$ defined by

$$\forall t \in I : O_U(y)(t) = O_{U(t,\cdot),t}(y) = \int_0^t U^T(t, s)y(s)ds.$$

The worst-case error for $U \in \mathscr{F}$ is defined as

$$\sigma(U, \ell) := \limsup_{t_1\to\infty} \sigma(U(t_1, \cdot), t_1, \ell).$$

Intuitively, an infinite horizon observer is just a collection of finite horizon observers, one for each time interval. It maps any output defined on some interval (finite or infinite) to an estimate of a component of the corresponding state trajectory. The worst case error of an infinite horizon observer represents the largest asymptotic error of estimating $\ell^T Fx(t)$ as $t \to \infty$.

The effect of applying an infinite horizon observer $U \in \mathscr{F}$ to an output $y \in L^2_{loc}([0, +\infty), \mathbb{R}^p)$ of the system (1) can be described as follows. Assume that y corresponds to some initial state x_0 and noises f and η such that

$$x_0 Q_0 x_0 + \int_0^{+\infty} f^T(t)Qf(t) + \eta^T(t)R\eta(t)dt \le 1.$$

The latter restriction can equivalently be stated as $(x_0, f|_{[0,t_1]}, \eta|_{[0,t_1]}) \in \mathscr{E}(t_1)$, $\forall t_1 > 0$. Assume that x is the state trajectory corresponding to y. Then $O_U(y)$ represents an estimate of $\ell^T Fx$ and the estimation error is bounded from above by $\sigma(U,\ell)$ in the limit, i.e. for every $\epsilon > 0$ there exists $T > 0$ such that for all $t > T$

$$\sigma(U,\ell) + \epsilon > (\ell^T F x(t) - \mathbf{O}_U(y)(t))^2$$

So far we have defined observers as linear maps mapping past outputs to state estimates. For practical purposes it is desirable that the observer is represented by a stable LTI system.

Definition 2: The observer $U \in \mathscr{F}$ can be represented by a stable linear system, if there exists $A_o \in \mathbb{R}^{r\times r}, B_o \in \mathbb{R}^{r\times p}, C_o \in \mathbb{R}^{1\times r}$ such that A_o is stable and for any $y \in L^2_{loc}(I)$, $I = [0,t_1]$, $t_1 > 0$ or $I = [0,+\infty)$, the estimate $\mathscr{O}_U(y)$ is the output of the LTI system below:

$$\dot{s}(t) = A_o s(t) + B_o y(t), \;\; s(0) = 0$$
$$\forall t \in I : \mathcal{O}_U(y)(t) = C_o s(t).$$

The system $\mathscr{O}_U = (A_o, B_o, C_o)$ is called a *dynamical observer* associated with U.

In addition, we would like to find observers with the smallest possible worst case observation error. These two considerations prompt us to define the minimax observer design problem as follows.

Problem 1 (Minimax observer design): Find an observer $\widehat{U} \in \mathscr{F}$ such that

$$\sigma(\widehat{U},\ell) = \inf_{U\in\mathscr{F}} \sigma(U,\ell) < +\infty \tag{3}$$

and $\widehat{U}$ can be represented by a stable linear system. In what follows we will refer to such $\widehat{U} \in \mathscr{F}$ as minimax observer and $\mathscr{O}_{\widehat{U}}$ as a dynamical minimax observer.

III. Main results

In this section we present our main result: minimax observer for the infinite horizon case. First, in §III-A we present the dual *optimal control problem* for infinite horizon case. In order to solve it, we will use the concept of output zeroing space from the geometric control. This technique allows us to construct an LTI system whose outputs are solutions of the original DAE (see §III-B). In §III-C we reformulate the dual optimal control problem as a linear quadratic infinite horizon control problem for LTIs. The solution of the latter yields a solution to the dual control problem. Finally, in §III-D we present the formulas for the minimax observer and discuss the conditions for its existence.

A. Dual control problem

We will start with formulating an optimal control problem for DAEs. Later on, we will show that the solution of this control problem yields a solution to the minimax observer design problem. Consider the DAE Σ:

$$\frac{dEx}{dt} = \hat{A}x(t) + \hat{B}u(t) \text{ and } Ex(0) = Ex_0. \tag{4}$$

Here $x_0 \in \mathbb{R}^n$ is a fixed initial state and $\hat{A}, E \in \mathbb{R}^{n\times n}$, $\hat{B} \in \mathbb{R}^{n\times m}$.

Notation 1 ($\mathscr{D}_{x_0}(t_1)$ and $\mathscr{D}_{x_0}(\infty)$): . For any $t_1 \in [0,+\infty]$ denote by I the interval $[0,t_1]\cap[0,+\infty)$ and denote by $\mathscr{D}_{x_0}(t_1)$ the set of all pairs $(x,u) \in L^2_{loc}(I,\mathbb{R}^n) \times L^2_{loc}(I,\mathbb{R}^m)$ such that Fx is absolutely continuous and (x,u) satisfy (4).

Note that we did not assume that the DAE is regular, and hence there may exist initial states x_0 such that $\mathcal{D}_{x_0}(t_1)$ is empty for some $t_1 \in [0,+\infty]$.

Problem 2 (Optimal control problem:): Take $R \in \mathbb{R}^{m\times m}, Q, Q_0 \in \mathbb{R}^{n\times n}$ and assume that $R > 0, Q > 0$ $Q_0 \geq 0$. For any initial state $x_0 \in \mathbb{R}^n$, and any trajectory $(x,u) \in \mathscr{D}_{x_0}(t)$, $t > t_1$ define the cost functional

$$\begin{aligned} J(x,u,t_1) &= x^T(t_1)E^T Q_0 E x(t_1) + \\ &+ \int_0^{t_1} (x^T(s)Qx(s) + u^T(s)Ru(s))ds. \end{aligned} \tag{5}$$

For every $(x,u) \in \mathscr{D}(\infty)$, define

$$J(x,u) = \limsup_{t_1\to\infty} J(x,u,t_1)\,.$$

The infinite horizon optimal control problem for (4) is the problem of finding a tuple of matrices (A_c, B_c, C_x, C_u) such that 1) $A_c \in \mathbb{R}^{r\times r}, B_c \in \mathbb{R}^{r\times n}$, 2) $C_x \in \mathbb{R}^{n\times r}$, $C_u \in \mathbb{R}^{m\times r}$, A_c is a stable matrix, 3) $B_c E C_x = I_r$, and 4) for any $x_0 \in \mathbb{R}^n$ such that $\mathcal{D}_{x_0}(\infty) \neq \emptyset$, the output of the system

$$\begin{aligned} \dot{s}(t) = A_c s(t) &\text{ and } s(0) = B_c E x_0 \\ x^*(t) = C_x s(t) &\text{ and } u^*(t) = C_u s(t), \end{aligned} \tag{6}$$

is such that $(x^*, u^*) \in \mathscr{D}_{x_0}(\infty)$, and

$$J(x^*,u^*) = \limsup_{t_1\to\infty} \inf_{(x,u)\in\mathscr{D}_{x_0}(t_1)} J(x,u,t_1). \tag{7}$$

The tuple $\mathscr{C}^* = (A_c, B_c, C_x, C_u)$ will be called the *dynamic controller*. For each x_0, the pair (x^*, u^*) will be referred as the solution of the optimal control problem for the initial state x_0.

We will denote infinite horizon control problems above by $\mathcal{C}(E, \hat{A}, \hat{B}, Q, R, Q_0)$.

Note that the dynamic controller which generates the solutions of the optimal control problem does not depend on the initial condition. In fact, the dynamical controller generates a solution for any initial condition x_0, for which the DAE (4) admits a solution on the whole time axis, i.e. for which $\mathscr{D}_{x_0}(\infty) \neq \varnothing$.

Remark 1 (Solution as feedback): In our case, the optimal control law u^* can be interpreted as a state feedback. Note, however, that for DAEs the feedback law does not determine the control input uniquely, since even autonomous DAEs may admit several solutions starting from the same initial state.

Now we are ready to present the relationship between Problem 2 and Problem 1.

Definition 3 (Dual control problem): The dual control problem for the observer design problem is the control problem $\mathcal{C}(F^T, A^T, -H^T, Q^{-1}, R^{-1}, \bar{Q}_0)$, where

$$\bar{Q}_0 = (F^{T^+} z(0) - \mathcal{M}_{opt})^T Q_0^{-1} (F^{T^+} z(0) - \mathcal{M}_{opt}).$$

Here $\mathcal{M}_{opt}$ is defined as follows. Let $r = \text{Rank}F^T$ and $U \in \mathbb{R}^{n\times(n-r)}$ such that $\text{im}U = \ker F^T$ and define $\mathcal{M}_{opt} = U(U^TQ_0^{-1}U)^{-1}U^TQ_0^{-1}F^{T+}$.

Theorem 1 (Duality): Let $\mathscr{C}_{u^*} = (A_c, B_c, C_x, C_u)$ be the dynamic controller solving the dual control problem. Let (x^*, u^*) be the corresponding solution of the optimal control problem for $x_0 = \ell$. Then $\widehat{U}(t_1, s) = u^*(t_1 - s)$ is the minimax observer design problem, and

$$\sigma(\widehat{U}, \ell) = J(x^*, u^*) = \limsup_{t_1\to\infty}\{x^{*T}(t_1)F\bar{Q}_0F^Tx^*(t_1)+ \int_0^{t_1}(u^{*T}(t)R^{-1}u^*(t) + x^{*T}(t)Q^{-1}x^*(t))dt\}.$$

In addition, the dynamic minimax observer $\mathscr{O}_{\widehat{U}}$ is of the form

$$\dot{s}(t) = A_c^Ts(t) + C_u^Ty(t),\ s(0) = 0$$
$$\mathcal{O}_{\hat{U}}(y)(t) = \ell^TFB_c^Ts(t)$$

Moreover, if $y \in L^2_{loc}([0,+\infty), \mathbb{R}^p)$ is the output of (1) for $f = 0$ and $\eta = 0$, then the estimation error $(\ell^TFx(t) - \mathcal{O}_{\hat{U}}(y)(t))$ converges to zero as $t \to \infty$.

The proof of Theorem 1 is omitted due lack of space, see [23]. It relies on [22, Proposition 1]. Note that the matrices of the observer presented in Theorem 1 depend on ℓ only through the equation $\mathcal{O}_{\hat{U}}(y)(t) = \ell^TFB_c^Ts(t)$. Hence, if a solution to the dual control problem exists, then it yields an observer for any ℓ, for which the adjoint DAE $\frac{d(F^Tz(t))}{dt} = A^Tz(t) - H^Tv(t), \quad F^Tz(0) = F^T\ell$ has a solution defined on the whole time axis.

Theorem 1 implies that existence of a solution of the dual control problem is a sufficient condition for existence of a solution for Problem 1. In fact, we conjecture that this condition is also a necessary one.

B. DAE systems as solutions to the output zeroing problem

Consider the DAE system (4). In this section we will study solution set $\mathscr{D}_{x_0}(t_1)$, $t_1 \in [0, +\infty]$ of (4). It is well known that for any fixed x_0 and u, (4) may have several solutions or no solution at all. In the sequel, we will use the tools of geometric control theory to find a subset $\mathcal{X}$ of $\mathbb{R}^n$, such that for any $x_0 \in E^{-1}(\mathcal{X})$, $\mathcal{D}_{x_0}(t_1) \neq \emptyset$ for all $t_1 \in [0, +\infty]$. Furthermore, we provide a complete characterization of all such solutions as outputs of an LTI system.

Theorem 2: Consider the DAE system (4). There exists a linear system $\mathscr{S} = (A_l, B_l, C_l, D_l)$ with $A_l \in \mathbb{R}^{\hat{n}\times\hat{n}}$, $B_l \in \mathbb{R}^{\hat{n}\times k}$, $C_l \in \mathbb{R}^{(n+m)\times\hat{n}}$ and $D_l \in \mathbb{R}^{(n+m)\times k}$, $\hat{n} \leq n$, and a linear subspace $\mathcal{X} \subseteq \mathbb{R}^n$ such that the following holds.

- $\text{Rank}D_l = k$.
- Consider the partitioning $C_l = \begin{bmatrix}C_s^T, & C_{inp}^T\end{bmatrix}^T$, $D_l = \begin{bmatrix}D_s^T, & D_{inp}^T\end{bmatrix}^T$, $C_s \in \mathbb{R}^{n\times\hat{n}}$, $C_{inp} \in \mathbb{R}^{m\times\hat{n}}$, $D_s \in \mathbb{R}^{n\times k}$, $D_{inp} \in \mathbb{R}^{m\times k}$. Then $ED_s = 0$, $\text{Rank}EC_s = \hat{n}$, $\mathcal{X} = \text{im}EC_s$.
- For any $t_1 \in [0, +\infty]$,
$$\mathcal{D}_{x_0}(t_1) \neq \emptyset \iff Ex_0 \in \mathcal{X}.$$
- Define the map $\mathcal{M} = (EC_s)^+ : \mathcal{X} \to \mathbb{R}^{\hat{n}}$. Then $(x, u) \in \mathcal{D}_{x_0}(t_1)$ for some $t_1 \in [0, +\infty]$ if and only if there exists an input $g \in L^2(I, \mathbb{R}^k)$, $I = [0, t_1] \cap [0, +\infty)$, such that
$$\dot{v} = A_lv + B_lg \text{ and } v(0) = \mathcal{M}(Ex_0)$$
$$x = C_sv + D_sg, \qquad .$$
$$u = C_{inp}v + D_{inp}g,$$
Moreover, in this case, the state trajectories x and v are related as $\mathcal{M}(Ex) = v$.

Proof: [Proof of Theorem 2] There exists suitable nonsingular matrices S and T such that

$$SET = \begin{bmatrix}I_r & 0\\ 0 & 0\end{bmatrix}, \tag{8}$$

where $r = \text{Rank}E$. Let

$$S\hat{A}T = \begin{bmatrix}\widetilde{A} & A_{12}\\ A_{21} & A_{22}\end{bmatrix},\ S\hat{B} = \begin{bmatrix}B_1\\ B_2\end{bmatrix}$$

be the decomposition of A, B such that $A_{11} \in \mathbb{R}^{r\times r}$, $B_{11} \in \mathbb{R}^{r\times m}$. Define

$$G = \begin{bmatrix}A_{12}, & B_1\end{bmatrix},\quad \widetilde{D} = \begin{bmatrix}A_{22}, & B_2\end{bmatrix} \text{ and } \widetilde{C} = A_{21}.$$

Consider the following linear system

$$\mathcal{S}\begin{cases}\dot{p} = \widetilde{A}p + Gq\\ z = \widetilde{C}p + \widetilde{D}q\end{cases}. \tag{9}$$

The trajectories (x, u) of the DAE (4) are exactly those trajectories (p, q), $T^{-1}x = (p^T, q_1^T)^T$, $q = (q_1^T, u^T)^T$, $q_1 \in \mathbb{R}^{n-r}$, of the linear system (9) for which the output z is zero.

Recall from [17, Section 7.3] the problem of making the output zero by choosing a suitable input. Recall from [17, Definition 7.8] the concept of a weakly observable subspace of a linear system. If we apply this concept to $\mathcal{S}$, then an initial state $p(0) \in \mathbb{R}^r$ of $\mathcal{S}$ is *weakly observable*, if there exists an input function $q \in L^2([0, +\infty), \mathbb{R}^k)$ such that the resulting output $z(t) = 0$ for all $t \in [0, +\infty)$. Following the convention of [17], let us denote the set of all weakly observable initial states by $\mathcal{V}(\mathcal{S})$. As it was remarked in [17, Section 7.3], $\mathcal{V}(\mathcal{S})$ is a vector space and in fact it can be computed. Moreover, if $p(0)$ in $\mathcal{V}(\mathcal{S})$ and for the particular choice of q, $z = 0$, then $p(t) \in \mathcal{V}(\mathcal{S})$ for all $t \geq 0$.

Let $I = [0, t]$ or $I = [0, +\infty)$. Let $q \in L^2(I, \mathbb{R}^{n-r+m})$ and let $p_0 \in \mathbb{R}^r$. Denote by $p(p_0, q)$ and $z(p_0, q)$ the state and output trajectory of (9) which corresponds to the initial state p_0 and input q. For technical purposes we will need the following easy extension of [17, Theorem 7.10–.11].

Theorem 3: 1) $\mathcal{V} = \mathcal{V}(\mathcal{S})$ is the largest subspace of $\mathbb{R}^r$ for which there exists a linear map $\widetilde{F} : \mathbb{R}^r \to \mathbb{R}^{m+n-r}$ such that

$$(\widetilde{A} + G\widetilde{F})\mathcal{V} \subseteq \mathcal{V} \text{ and } (\widetilde{C} + \widetilde{D}\widetilde{F})\mathcal{V} = 0 \tag{10}$$

2) Let $\widetilde{F}$ be a map such that (10) holds for $\mathcal{V} = \mathcal{V}(\mathcal{S})$. Let $L \in \mathbb{R}^{(m+n-r)\times k}$ for some k be a matrix such that $\text{im}L = \ker\widetilde{D} \cap G^{-1}(\mathcal{V}(\mathcal{S}))$ and $\text{Rank}L = k$.

For any interval $I = [0, t]$ or $I = [0, +\infty)$, and for any $p_0 \in \mathbb{R}^r$, $q \in L^2_{loc}(I, \mathbb{R}^k)$,

$$z(p_0, q)(t) = 0 \text{ for } t \in I \text{ a.e.}$$

if and only if $p_0 \in \mathcal{V}$ and there exists $w \in L^2_{loc}(I, \mathbb{R}^{n-r+m})$ such that

$$q(t) = \widetilde{F} p(p_0, q)(t) + Lw(t) \text{ for } t \in I \text{ a.e.}$$

We are ready now to finalize the proof of Theorem 2. The desired linear system $\mathscr{S} = (A_l, B_l, C_l, D_l)$ is now obtained as follows. Consider the linear system:

$$\begin{aligned}
&\dot{p} = (\widetilde{A} + G\widetilde{F})p + GLw \\
&(x^T, u^T)^T = \bar{C}p + \bar{D}w \\
&\bar{C} = \begin{bmatrix} T & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} I_r \\ \widetilde{F} \end{bmatrix} \text{ and } \bar{D} = \begin{bmatrix} T & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} 0 \\ L \end{bmatrix}.
\end{aligned}$$

Choose a basis of $\mathcal{V} = \mathcal{V}(\mathcal{S})$ and choose (A_l, B_l, C_l, D_l) as follows: $D_l = \bar{D}$, and let A_l, B_l, C_l be the matrix representations in this basis of the linear maps $(\widetilde{A} + G\widetilde{F}) : \mathcal{V} \to \mathcal{V}$, $GL : \mathbb{R}^k \to \mathcal{V}$, and $\bar{C} : \mathcal{V} \to \mathbb{R}^{n+m}$ respectively. Define

$$\mathcal{X} = \{S^{-1} \begin{bmatrix} p \\ 0 \end{bmatrix} \mid p \in \mathcal{V}\}.$$

It is easy to see that this choice of (A_l, B_l, C_l, D_l) and $\mathcal{X}$ satisfies the conditions of the theorem. ∎

The proof of Theorem 2 is constructive and yields an algorithm for computing (A_l, B_l, C_l, D_l) from $(E, \hat{A}, \hat{B})$. This prompts us to introduce the following terminology.

Definition 4: A linear system $\mathscr{S} = (A_l, B_l, C_l, D_l)$ described in the proof of Theorem 2 is called the linear system associated with the DAE (4).

Note that the linear system associated with $(E, \hat{A}, \hat{B})$ is not unique. However, we can show that all associated linear systems are feedback equivalent, see [23] for details.

C. Solution of the optimal control problem for DAE

We apply Theorem 2 in order to solve a control problem defined in Problem 2. Let $\mathscr{S} = (A_l, B_l, C_l, D_l)$ be a linear system associated with Σ and let $\mathcal{M}$ be the map described in Theorem 2 and let C_s be the component of C_l as defined in Theorem 2. Consider the following linear quadratic control problem. For every initial state v_0, for every interval I containing $[0, t_1]$ and for every $g \in L^2_{loc}(I, \mathbb{R}^k)$ define the cost functional $J(v_0, g, t)$

$$\begin{aligned}
&\mathscr{J}(v_0, g, t_1) = v^T(t_1) E^T C_s^T Q_0 E C_s v(t_1) + \\
&+ \int_0^{t_1} \nu^T(t) \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \nu(t) dt \\
&\dot{v} = A_l v + B_l g \text{ and } v(0) = v_0 \\
&\nu = C_l v + D_l g.
\end{aligned}$$

For any $g \in L^2_{loc}([0, +\infty), \mathbb{R}^k)$ and $v_0 \in \mathbb{R}^{\hat{n}}$, define

$$\mathscr{J}(v_0, g) = \limsup_{t_1 \to \infty} \mathscr{J}(v_0, g, t_1).$$

Consider the control problem of finding for every initial state v_0 an input $g^* \in L^2_{loc}(\mathbb{R}^k)$ such that

$$\mathscr{J}(v_0, g^*) = \limsup_{t_1 \to \infty} \inf_{g \in L^2(0, t_1)} \mathscr{J}(v_0, g, t_1). \tag{11}$$

Definition 5 (Associated LQ problem): The control problem (11) is called an *LQ problem associated* with $\mathcal{C}(E, \hat{A}, \hat{B}, Q, R, Q_0)$ and it is denoted by $\mathcal{CL}(A_l, B_l, C_l, D_l)$.

The relationship between the associated LQ problem and the original control problem for DAEs is as follows.

Theorem 4: Let $g^* \in L^2_{loc}([0, +\infty), \mathbb{R}^k)$ and let (x^*, u^*) be the corresponding output of $\mathscr{S} = (A_l, B_l, C_l, D_l)$ from the initial state $v_0 = \mathcal{M}(Ex_0)$ for some $x_0 \in \mathbb{R}^n$, $\mathscr{D}_{x_0}(\infty) \neq \emptyset$. Then $(x^*, u^*) \in \mathscr{D}_{x_0}(\infty)$ and g^* is a solution of $\mathcal{CL}(A_l, B_l, C_l, D_l)$ for v_0 if and only if

$$J(x^*, u^*) = \limsup_{t_1 \to \infty} \inf_{(x,u) \in \mathscr{D}_{x_0}(t_1)} J(x, u, t).$$

The proof of Theorem 4 can be found in [23].

The solution of associated LQ problem can be derived using classical results, see [10].

Theorem 5: Let $\mathcal{CL}(A_l, B_l, C_l, D_l)$ be the LQ problem associated with $\mathcal{C}(E, \hat{A}, \hat{B}, Q, R, Q_0)$. Assume that (A_l, B_l) is stabilizable. Define $S = \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}$. Consider the algebraic Riccati equation

$$\begin{aligned}
&0 = PA_l + A_l^T P - K^T (D_l^T S D_l) K + C_l^T S C_l. \\
&K = (D_l^T S D_l)^{-1} (B_l^T P + D_l^T S C_l).
\end{aligned} \tag{12}$$

Then (12) has a unique solution $P > 0$ such that $A_l - B_l K$ is a stable matrix. Moreover, if g^* is defined as

$$\begin{aligned}
&\dot{v}^* = A_l v^* + B_l g^* \text{ and } v^*(0) = v_0 \\
&g^* = -Kv^*,
\end{aligned} \tag{13}$$

then g^* is a solution of $\mathcal{CL}(A_l, B_l, C_l, D_l)$ for the initial state v_0 and $v_0^T P v_0 = \mathscr{J}(v_0, g^*)$.

Combining Theorem 5 and Theorem 4, we can solve the optimal control problem for DAEs as follows.

Corollary 1: Consider the control problem $\mathcal{C}(E, \hat{A}, \hat{B}, Q, R, Q_0)$ and let $\mathcal{CL}(A_l, B_l, C_l, D_l)$ be an LQ problem associated with $\mathcal{C}(E, \hat{A}, \hat{B}, Q, R, Q_0)$. Assume that (A_l, B_l) is stabilizable. Let P be the unique positive definite solution of (12) and let K be as in (12). Let $C_s, C_{inp}, D_s, D_{inp}$ be the decomposition of C_l and D_l as defined in Theorem 2 and let $\mathcal{M} = (EC_s)^+$. Then the dynamical controller $\mathscr{C} = (A_c, B_c, C_x, C_u)$ with

$$\begin{aligned}
&A_c = A_l - B_l K, \; C_x = C_s - D_s K \\
&C_u = (C_{inp} - D_{inp} K) \text{ and } B_c = \mathcal{M}.
\end{aligned}$$

is a solution of $\mathcal{C}(E, \hat{A}, \hat{B}, Q, R, Q_0)$.

Remark 2 (Computation and existence of a solution): A solution for Problem 2 can be computed from the matrices $(E, \hat{A}, \hat{B}, Q, R, Q_0)$. Notice that the only condition for the existence of a solution is that the associated linear system $\mathscr{S} = (A_l, B_l, C_l, D_l)$ is stabilizable. Since all linear systems associated with the given DAE are feedback equivalent, stabilizability of an associated linear system does not depend on the choice of the linear system. Thus, stabilizability of $\mathscr{S}$ can be regarded as a property of $(E, \hat{A}, \hat{B})$. The link between stabilizability of $\mathscr{S}$ and the classical stabilizability for DAEs remains a topic for future research.

D. Observer design for DAE

By applying Corollary 1 and Theorem 1, we obtain the following procedure for solving Problem 1.

- **Step 1.** Consider the dual DAE of the form (4), such that $F^T = E$, $A^T = \hat{A}$ and $-H^T = \hat{B}$. Construct a linear system $\mathscr{S} = (A_l, B_l, C_l, D_l)$ associated with this DAE, as described in Definition 4.
- **Step 2.** Check if (A_l, B_l) is stabilizable. If it is, let

$$X = \begin{bmatrix} Q^{-1} & 0 \\ 0 & R^{-1} \end{bmatrix}.$$

  Consider the algebraic Riccati equation

$$\begin{aligned} 0 &= PA_l + A_l^T P - K^T (D_l^T X D_l) K + C_l^T X C_l. \\ K &= (D_l^T X D_l)^{-1} (B_l^T P + D_l^T X C_l). \end{aligned} \tag{14}$$

  The equation (14) has a unique solution $P > 0$.
- **Step 3.** The dynamical observer $\mathscr{O}_{\widehat{U}}$ which is a solution of Problem 1 is of the form:

$$\begin{aligned} \dot{r}(t) &= (A_l - B_l K)^T r(t) + (C_l - D_l K)^T \begin{bmatrix} 0 \\ y(t) \end{bmatrix} \\ \mathcal{O}_{\widehat{U}}(y)(t) &= \ell^T F \mathcal{M}^T r(t), \end{aligned}$$

  and $\widehat{U}(t, s) = (C_l - D_l K) e^{(A_l - B_l K)(t-s)} \mathcal{M} F^T \ell$. The observation error equals

$$\sigma(\widehat{U}, \ell) = \ell^T F \mathcal{M}^T P \mathcal{M} F^T \ell.$$

  Recall that $\mathcal{M} = (F^T C_s)^+$, where C_s is the submatrix of C_l formed by its first n rows.

Remark 3 (Conditions for existence of an observer): The existence of the observer above depends only on whether the chosen linear system associated with the dual DAE is stabilizable. The latter could be thought of as a sort of detectability property. The relationship between this property and the notion of detectability which is established in the literature remains a topic of future research.

IV. Conclusions

We have presented a solution to the minimax observer design problem and the infinite horizon linear quadratic control problem for linear DAEs. We have also shown that these two problems are each other's dual. The main novelty of this contribution is that we made no solvability assumptions on DAEs. The only condition we need is that the LTI associated with the dual DAE should be stabilizable. We conjecture that this condition is also a necessary one. The clarification of this issue remains a topic of future research.

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