Parameter identification and state estimation for linear parabolic equations

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Outline

Minimax projection method
- Projection coefficients as a solution of DAE
- Bounding set for the projection error
- Ellipsoid containing the projection coefficients
- State estimation for a linear transport equation
- Parameter identification for linear Darcy equation
Problem statement

Assume $a > 0$ and $I(\cdot, t) \in H_0^1(\Omega)$ satisfies for almost all $t \in (0, T)$ the following equation:

$$\partial_t I + \mathbf{M} \cdot \nabla I - a \Delta I = f, \quad I(x, 0) = f_0(x),$$

where

- $\mathbf{x} \in \Omega \subset \mathbb{R}^n$, $n \geq 2$, $\Omega$ is an open bounded convex set;
- $\mathbf{M}(\mathbf{x}, t) = (M_1(\mathbf{x}, t) \ldots M_n(\mathbf{x}, t))^\prime$ with $M_i \in L^\infty(0, T, H_0^1(\Omega))$ for all $i = 1, \ldots, n$;
- $f \in L^2(0, T, L^2(\Omega))$ and $f_0 \in H^2(\Omega) \cap H_0^1(\Omega)$. 

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Estimation and identification for parabolic PDEs (Sergiy Zhuk) NDNS+ workshop, EMI
We expand the solution $I$ into the following series:

$$I(x, t) = \sum_{i \in \mathbb{N}} a_i(t) \varphi_i(x), \quad a_i(t) := \langle I(\cdot, t), \varphi_i \rangle_{L^2(\Omega)},$$

(1)

where $\{\varphi_k\}_{k \in \mathbb{N}}$ is the orthonormal set of eigenfunctions of $-\Delta$:

$$-\Delta \varphi_k = \lambda_k \varphi_k, \quad \varphi_k \in C^\infty(\Omega) \cap H^1_0(\Omega), \quad \varphi_k = 0 \text{ on } \partial\Omega.$$
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$$ I(x, t) = \sum_{i \in \mathbb{N}} a_i(t) \varphi_i(x), \quad a_i(t) := \langle I(\cdot, t), \varphi_i \rangle_{L^2(\Omega)}, \quad (1) $$

where $\{\varphi_k\}_{k \in \mathbb{N}}$ is the orthonormal set of eigenfunctions of $-\Delta$:

$$ -\Delta \varphi_k = \lambda_k \varphi_k, \quad \varphi_k \in C^\infty(\Omega) \cap H^1_0(\Omega), \quad \varphi_k = 0 \text{ on } \partial \Omega. $$

Define projection operator:

$$ P_N I(\cdot, t) = a(t) := (a_1(t) \ldots a_N(t))^\prime, $$

and reconstruction operator:

$$ P_N^+ a(t) = \sum_{i=1}^{N} a_i(t) \varphi_i $$
We expand the solution $I$ into the following series:

$$I(x, t) = \sum_{i \in \mathbb{N}} a_i(t) \varphi_i(x), \quad a_i(t) := \langle I(\cdot, t), \varphi_i \rangle_{L^2(\Omega)} \quad \text{(1)}$$

where $\{\varphi_k\}_{k \in \mathbb{N}}$ is the orthonormal set of eigenfunctions of $-\Delta$:

$$-\Delta \varphi_k = \lambda_k \varphi_k, \quad \varphi_k \in C^\infty(\Omega) \cap H^1_0(\Omega), \quad \varphi_k = 0 \text{ on } \partial \Omega.$$ 

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$$P_N^+ a(t) = \sum_{i=1}^{N} a_i(t) \varphi_i$$

and the vector of the exact projection coefficients: $a_N^{\text{true}} := P_N I$. 

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**Galerkin projection**

Estimation and identification for parabolic PDEs (Sergiy Zhuk) NDNS+ workshop, EMI
Define a differential operator \( A \varphi = \mathbf{M} \cdot \nabla \varphi - a \Delta \varphi \) and a commutation error:

\[
e(x, t) := A P_N^+ P_N l(x, t) - P_N^+ P_N A l(x, t).
\]  

(2)
Define a differential operator $A \varphi = M \cdot \nabla \varphi - a \Delta \varphi$ and a commutation error:

$$e(x, t) := AP_N^+ P_N l(x, t) - P_N^+ P_N A l(x, t).$$  \hspace{1cm} (2)$$

Since $a_N^{\text{true}}(t) = P_N l(\cdot, t)$ it follows that $a_N^{\text{true}}$ solves:

$$\partial_t P_N^+ a = P_N^+ P_N \partial_t l = -A P_N^+ a + e + P_N^+ P_N f.$$  \hspace{1cm} (3)$$
Define a differential operator $A\varphi = M \cdot \nabla \varphi - a \Delta \varphi$ and a commutation error:

$$e(x, t) := AP_N^+ P_N I(x, t) - P_N^+ P_N A I(x, t).$$ \hfill (2)

Since $a_N^{\text{true}}(t) = P_N I(\cdot, t)$ it follows that $a_N^{\text{true}}$ solves:

$$\partial_t P_N^+ a = P_N^+ P_N \partial_t I = -AP_N^+ a + e + P_N^+ P_N f.$$ \hfill (3)

As $P_N P_N^+ = I$, we get, multiplying (3) by $P_N$, that $a_N^{\text{true}}$ solves

$$\frac{da}{dt} = -P_N A P_N^+ a + P_N e + P_N f$$
On the other hand,

\[ \partial_t P_N^+ a = P_N^+ P_N \partial_t l = -A P_N^+ a + e + P_N^+ P_N f \]

has a solution if and only if \(-A P_N^+ a + e\) is in the range of \(P_N^+\).
DAE for the projection coefficients

On the other hand,

$$\partial_t P_N^+ a = P_N^+ P_N \partial_t I = -A P_N^+ a + e + P_N^+ P_N f$$

has a solution if and only if $-A P_N^+ a + e$ is in the range of $P_N^+$. This holds true, in turn, if

$$(\mathcal{I} - P_N^+ P_N) A P_N^+ a = (\mathcal{I} - P_N^+ P_N) e.$$
On the other hand,
\[ \partial_t P^+_N a = P^+_N P_N \partial_t l = -A P^+_N a + e + P^+_N P_N f \] (4)
has a solution if and only if \(-A P^+_N a + e\) is in the range of \(P^+_N\).
This holds true, in turn, if
\[ (I - P^+_N P_N) A P^+_N a = (I - P^+_N P_N) e. \]
Noting that
\[ (I - P^+_N P_N) e(t) = (I - P^+_N P_N) A P^+_N a^\text{true}_N \]
and, recalling that \((P^+_N)’ = P_N\), we compute:
\[ \| (I - P^+_N P_N) A P^+_N a^\text{true}_N \|_{L^2(\Omega)}^2 = (S_N - A'_N A_N) a^\text{true}_N \cdot a^\text{true}_N = \| H_N a^\text{true}_N \|_{\mathbb{R}^N}^2, \]
where \(S_N = \{ \langle A_\varphi_i, A_\varphi_j \rangle \}_{i,j=1}^N\), \(A_N = P_N A P^+_N\) and
\[ H_N := (S_N - A'_N A_N)^{\frac{1}{2}}. \]
DAE for the projection coefficients

On the other hand,

\[ \partial_t \mathbf{P}_N^+ \mathbf{a} = \mathbf{P}_N^+ \mathbf{P}_N \partial_t \mathbf{l} = -A \mathbf{P}_N^+ \mathbf{a} + \mathbf{e} + \mathbf{P}_N^+ \mathbf{P}_N f \]  

has a solution if and only if \(-A \mathbf{P}_N^+ \mathbf{a} + \mathbf{e}\) is in the range of \(\mathbf{P}_N^+\). This holds true, in turn, if

\[ (\mathcal{I} - \mathbf{P}_N^+ \mathbf{P}_N) A \mathbf{P}_N^+ \mathbf{a} = (\mathcal{I} - \mathbf{P}_N^+ \mathbf{P}_N) \mathbf{e}. \]

Noting that

\[ (\mathcal{I} - \mathbf{P}_N^+ \mathbf{P}_N) \mathbf{e}(t) = (\mathcal{I} - \mathbf{P}_N^+ \mathbf{P}_N) A \mathbf{P}_N^+ \mathbf{a}_N^{\text{true}} \]

and, recalling that \((\mathbf{P}_N^+)' = \mathbf{P}_N\), we compute:

\[ \| (\mathcal{I} - \mathbf{P}_N^+ \mathbf{P}_N) A \mathbf{P}_N^+ \mathbf{a}_N^{\text{true}} \|_{L^2(\Omega)}^2 = (\mathbf{S}_N - \mathbf{A}_N^T \mathbf{A}_N) \mathbf{a}_N^{\text{true}} \cdot \mathbf{a}_N^{\text{true}} = \| \mathbf{H}_N \mathbf{a}_N^{\text{true}} \|_{\mathbb{R}^N}^2, \]

where \(\mathbf{S}_N = \{ \langle A \varphi_i, A \varphi_j \rangle \}_{i,j=1}^N\), \(\mathbf{A}_N = \mathbf{P}_N A \mathbf{P}_N^+\) and

\(\mathbf{H}_N := (\mathbf{S}_N - \mathbf{A}_N^T \mathbf{A}_N)^{\frac{1}{2}}\).

Thus \(\mathbf{a}_N^{\text{true}}\) solves the algebraic equation:

\[ 0 = \mathbf{H}_N \mathbf{a} + \mathbf{e}^{\circ} \]

for \(\mathbf{e}^{\circ} = -\mathbf{H}_N \mathbf{a}_N^{\text{true}}\).
Finally we find that if $\partial_t l + A l = f$, $l(0) = f_0$ then $a^{\text{true}}_N = P_N^+ P_N^+ l$ solves the following DAE:

\[
\frac{da}{dt} = -P_N A P_N^+ a + e^m + P_N f, \\
0 = H_N a + e^o, a(0) = P_N f_0, \\
e^m = P_N e = P_N A (P_N^+ P_N l - l) \\
e^o = -H_N P_N l
\]  

where $S_N = \{ \langle A \phi_i, A \phi_j \rangle \}_{i,j=1}^N$ and $H_N := (S_N - A_N' A_N)^{\frac{1}{2}}$. 
Minimax projection method
Projection coefficients as a solution of DAE
Bounding set for the projection error
Ellipsoid containing the projection coefficients
State estimation for a linear transport equation
Parameter identification for linear Darcy equation
A priori estimates

For $A\varphi = M \cdot \nabla \varphi - a\Delta \varphi$ we get an estimate:

$$e^m \cdot e^m = \|P_N A(P_N^+ P_N l - l)\|_{\mathbb{R}^N}^2 = \sum_{k=1}^{N} \langle \varphi_k, M \cdot \nabla (P_N^+ P_N l - l) \rangle_{L^2(\Omega)}^2$$

$$\leq \|\rho_1(\cdot, t)\|_{L^\infty(\Omega)} \|\nabla (P_N^+ P_N l - l)\|_{L^2(\Omega)}^2$$

$$\leq \|\rho_1(\cdot, t)\|_{L^\infty(\Omega)} \lambda_{N+1}^{-1} \|\Delta l(\cdot, t)\|_{L^2(\Omega)}^2$$

where $\rho_1(x, t) := \|M(x, t)\|_{\mathbb{R}^n}^2$. 
For \( A\varphi = \mathbf{M} \cdot \nabla \varphi - a \Delta \varphi \) and \( I^N := P_N^+ P_N I \) we get an estimate:

\[
e^o \cdot e^o = \| H_N a_N^{\text{true}} \|_{\mathbb{R}^N}^2 = \|(I - P_N^+ P_N) A P_N^+ P_N I(\cdot, t)\|_{L^2(\Omega)}
\]

\[
= \sum_{k > N} \langle \varphi_k, A P_N^+ P_N I \rangle_{L^2(\Omega)}^2 = \sum_{k > N} \langle \varphi_k, \mathbf{M} \cdot \nabla P_N^+ P_N I \rangle_{L^2(\Omega)}^2
\]

\[
= \sum_{k > N} \lambda_k^{-2} \langle -\Delta \varphi_k, \mathbf{M} \cdot \nabla P_N^+ P_N I \rangle_{L^2(\Omega)}^2
\]

\[
\leq 2\lambda_{N+1}^{-1} \lambda_1^{-1} \rho_2(\cdot, t) + \rho_1(\cdot, t) \| \Delta I(\cdot, t) \|_{L^2(\Omega)}^2
\]

where \( \rho_1(\mathbf{x}, t) := \| \mathbf{M}(\mathbf{x}, t) \|_{\mathbb{R}^n}^2, \rho_2(\mathbf{x}, t) := \| J_{\mathbf{M}}(\mathbf{x}, t) \|^2, J_{\mathbf{M}} \) is the Jacobian of \( \mathbf{M} \).
Outline

Minimax projection method
  Projection coefficients as a solution of DAE
  Bounding set for the projection error
  Ellipsoid containing the projection coefficients
  State estimation for a linear transport equation
  Parameter identification for linear Darcy equation
Finally we find that if $\partial_t I + AI = f$, $I(0) = f_0$ then $a^\text{true}_N = P_N^+ P_N^+ l$ solves the following DAE:

$$\frac{d a}{d t} = -P_N A P_N^+ a + e^m + P_N f,$$

$$0 = H_N a + e^o, \quad a(0) = P_N f_0,$$

$$e^m = P_N e = P_N A (P_N^+ P_N l - l) \quad e^o = -H_N P_N l$$

and

$$\lambda_{N+1} \int_0^T \|e^m\|_{\mathbb{R}^N}^2 + \|e^o\|_{\mathbb{R}^N}^2 dt \leq C \int_0^T \|\Delta I(\cdot, t)\|_{L^2(\Omega)}^2 dt$$

$$\leq C_1 (\|\nabla f_0\|_{L^2(\Omega)}^2 + \int_0^T \|f(x, t)\|_{L^2(\Omega)}^2 dt)$$

$$\leq C^*$$

where $C^* = C^*(M, f_0, f)$ is a constant.
Ellipsoid for the projection coefficients

Let \( \hat{a} \) solve the following ODE:

\[
\frac{d\hat{a}}{dt} = - P_N A P_N^+ \hat{a} - \frac{\lambda_{N+1}}{C^*} K(H_N)'H_N \hat{a} + P_N f, \; \hat{a}(0) = P_N f_0 ,
\]

(7)

where \( C^* = C^*(M, f_0, f) \) is a constant, \( K = VU^{-1} \) and the matrix-valued functions \( V, U \) solve the following linear Hamiltonian ODE:

\[
\dot{U} = (P_N A P_N^+)'U + \frac{\lambda_{N+1}}{C^*} (H_N)'H_N V, \; U(0) = I ,
\]

(8)

\[
\dot{V} = -P_N A P_N^+ V + \frac{C^*}{\lambda_{N+1}} U, \; V(t_0) = 0 .
\]
Ellipsoid for the projection coefficients

Let $\hat{a}$ solve the following ODE:

$$\frac{d\hat{a}}{dt} = -P_NAP_N^+\hat{a} - \frac{\lambda_{N+1}}{C^*}K(H_N)'H_N\hat{a} + P_Nf, \hat{a}(0) = P_Nf_0,$$

(7)

where $C^* = C^*(M, f_0, f)$ is a constant, $K = VU^{-1}$ and the matrix-valued functions $V, U$ solve the following linear Hamiltonian ODE:

$$\dot{U} = (P_NAP_N^+)’U + \frac{\lambda_{N+1}}{C^*}(H_N)'H_NV, U(0) = I,$$

(8)

$$\dot{V} = -P_NAP_N^+V + \frac{C^*}{\lambda_{N+1}}U, V(t_0) = 0.$$

Assume that $l$ solves $\partial_t l + Al = f, l(0) = f_0.$
Ellipsoid for the projection coefficients

Let $\hat{a}$ solve the following ODE:

$$\frac{d\hat{a}}{dt} = - P_N A P_N^+ \hat{a} - \frac{\lambda_{N+1}}{C^*} K(H_N)'H_N \hat{a} + P_N f, \hat{a}(0) = P_N f_0,$$

(7)

where $C^* = C^*(M, f_0, f)$ is a constant, $K = VU^{-1}$ and the matrix-valued functions $V, U$ solve the following linear Hamiltonian ODE:

$$\dot{U} = (P_N A P_N^+)U + \frac{\lambda_{N+1}}{C^*} (H_N)'H_N V, U(0) = I,$$

$$\dot{V} = -P_N A P_N^+ V + \frac{C^*}{\lambda_{N+1}} U, V(t_0) = 0.$$  

(8)

Assume that $I$ solves $\partial_t I + AI = f, I(0) = f_0.$

Then

$$\|K^{-\frac{1}{2}}(t)(P_N I(\cdot, t) - \hat{a}(t))\|^2 \leq 1$$
Outline

Minimax projection method
- Projection coefficients as a solution of DAE
- Bounding set for the projection error
- Ellipsoid containing the projection coefficients

State estimation for a linear transport equation
- Parameter identification for linear Darcy equation
Problem statement

We assume that \( I \) solves the following linear hyperbolic equation:
\[
\frac{\partial}{\partial t} I + u \frac{\partial}{\partial x} I + v \frac{\partial}{\partial y} I = 0,
\]
\[
I(x, y, 0) = I_0(x, y), I = 0 \text{ on } \partial \Omega,
\]
where \( \Omega = (0, 2\pi)^2 \) and the fluid flow \( \mathbf{M} = (u(x, y, t), v(x, y, t))' \) is modelled by:
\[
\frac{\partial}{\partial t} \omega + u \frac{\partial}{\partial x} \omega + v \frac{\partial}{\partial y} \omega = 0,
\]
\[
u = -\frac{\partial}{\partial y} \psi, \quad v = \frac{\partial}{\partial x} \psi,
\]
\[- \Delta \psi = \omega, \psi(x, y) = 0, (x, y) \in \partial \Omega,
\]
\[
\omega(x, y, 0) = \omega_0(x, y), \omega(x, y, t) = 0 \text{ on } \partial \Omega.
\]

We aim at solving the following problem: given incomplete sparse observations of \( I \) estimate \( I \).
Numerical experiment: setup

- 75x75 basis functions $\varphi_{ks} := \sin\left(\frac{kx}{2}\right)\sin\left(\frac{sy}{2}\right)$ to represent vorticity and advected quantity $I$;
Numerical experiment: setup

- 75x75 basis functions \( \varphi_{ks} := \sin\left(\frac{kx}{2}\right)\sin\left(\frac{sy}{2}\right) \) to represent vorticity and advected quantity \( I \);
- strongly occluded observations every 20 timesteps;
\[ \partial_t l + u \partial_x l + v \partial_y l = e \]

\[ \partial_t \omega + u \partial_x \omega + v \partial_y \omega = f \]
\[ u = -\partial_y \psi, \quad v = \partial_x \psi, \quad -\Delta \psi = \omega \]
\[ \partial_t I + u \partial_x I + v \partial_y I = e \]

\[ \partial_t \omega + u \partial_x \omega + v \partial_y \omega = f \]

\[ u = -\partial_y \psi, \quad v = \partial_x \psi, \quad -\Delta \psi = \omega \]
\[ \partial_t l + u \partial_x l + v \partial_y l = e \]

\[ \partial_t \omega + u \partial_x \omega + v \partial_y \omega = f \]

\[ u = -\partial_y \psi, \ v = \partial_x \psi, \ -\Delta \psi = \omega \]
\[ \partial_t I + u \partial_x I + v \partial_y I = e \]

\[ \partial_t \omega + u \partial_x \omega + v \partial_y \omega = f \]

\[ u = -\partial_y \psi, \quad v = \partial_x \psi, \quad -\Delta \psi = \omega \]
\[ \partial_t I + u \partial_x I + v \partial_y I = e \]

\[ \partial_t \omega + u \partial_x \omega + v \partial_y \omega = f \]

\[ u = -\partial_y \psi, \quad v = \partial_x \psi, \quad -\Delta \psi = \omega \]
\[ \partial_t I + u \partial_x I + v \partial_y I = e \]

\[ \partial_t \omega + u \partial_x \omega + v \partial_y \omega = f \]

\[ u = -\partial_y \psi, \quad v = \partial_x \psi, \quad -\Delta \psi = \omega \]
$$\partial_t I + u \partial_x I + v \partial_y I = e$$

$$\partial_t \omega + u \partial_x \omega + v \partial_y \omega = f$$

$$u = -\partial_y \psi, \quad v = \partial_x \psi, \quad -\Delta \psi = \omega$$
\[ \partial_t l + u \partial_x l + v \partial_y l = e \]

\[ \partial_t \omega + u \partial_x \omega + v \partial_y \omega = f \]

\[ u = -\partial_y \psi, \ v = \partial_x \psi, \ -\Delta \psi = \omega \]

Ground truth
\[ \partial_t I + u \partial_x I + v \partial_y I = e \]

\[ \partial_t \omega + u \partial_x \omega + v \partial_y \omega = f \]

\[ u = -\partial_y \psi, \quad v = \partial_x \psi, \quad -\Delta \psi = \omega \]
Observations

Truth

Observed data
Observations

Truth

Observed data
Observations

Truth

Observed data
Observations

Truth

Observed data
Observations

Truth

Observed data
Reconstruction results

Truth

Estimate

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Estimation and identification for parabolic PDEs (Sergiy Zhuk) NDNS+ workshop, EMI
Reconstruction results

Truth

Estimate
Reconstruction results

Truth

Estimate

Estimation and identification for parabolic PDEs (Sergiy Zhuk)
Reconstruction results

Truth

Estimate
Reconstruction results

Truth

Estimate
Reconstruction results
Reconstruction results

Truth

Estimate
Reconstruction results

Truth

Estimate
Reconstruction results

Truth

Estimate

Estimation and identification for parabolic PDEs (Sergiy Zhuk)
Worst-case estimation error

Observation noise pattern

Worst-case error pattern
Relative observation error
Estimation of the projection coefficients
Outline

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Problem statement

Let \( h(\cdot, t) \in H^1_0(\Omega) \) solve the following linear parabolic PDE:

\[
\partial_t h = \partial_x (u \partial_x h) + \partial_y (u \partial_y h) + W, \\
h(0, x, y) = h_0(x, y), \quad h(t, 0, y) = h(t, a, y) = 0, \\
\partial_y h(t, x, 0) = \partial_y h(t, x, b) = 0.
\]

where \( \Omega := [0, a] \times [0, b] \)
Problem statement

Let $h(\cdot, t) \in H^1_0(\Omega)$ solve the following linear parabolic PDE:

$$
\begin{align*}
\partial_t h &= \partial_x(u \partial_x h) + \partial_y(u \partial_y h) + W, \\
h(0, x, y) &= h_0(x, y), h(t, 0, y) = h(t, a, y) = 0, \\
\partial_y h(t, x, 0) &= \partial_y h(t, x, b) = 0.
\end{align*}
$$

where $\Omega := [0, a] \times [0, b]$ and $Y_{kl}$ is observed in the following form:

$$
Y_{kl}(t_s) = \int_\Omega g_{kl}(x, y) h(t_s, x, y) \, dx \, dy + \eta_{s, kl},
$$

where $g_{kl} \in L^2(\Omega)$ is an averaging kernel supported in a point $(x_k, y_l) \in \Omega$ and $\eta_{kl} \in L^2(t_0, T)$ is an observation error.
Problem statement

Let \( h(\cdot, t) \in H^1_0(\Omega) \) solve the following linear parabolic PDE:
\[
\partial_t h = \partial_x(u \partial_x h) + \partial_y(u \partial_y h) + W, \\
h(0, x, y) = h_0(x, y), h(t, 0, y) = h(t, a, y) = 0, \\
\partial_y h(t, x, 0) = \partial_y h(t, x, b) = 0.
\]
where \( \Omega := [0, a] \times [0, b] \) and \( Y_{kl} \) is observed in the following form:
\[
Y_{kl}(t_s) = \int_{\Omega} g_{kl}(x, y) h(t_s, x, y) \, dx \, dy + \eta_{kl}^s, k = 1, p_x, l = 1, p_y,
\]
where \( g_{kl} \in L^2(\Omega) \) is an averaging kernel supported in a point \((x_k, y_l) \in \Omega\) and \( \eta_{kl} \in L^2(t_0, T) \) is an observation error.

We aim at solving the following problem:
\[
\sum_{k,l,s=1}^{p_x,p_y,M} R_{kl}(Y_{kl}(t_s) - \int_{\Omega} g_{kl}(x, y) h(t_s, x, y) \, dx \, dy)^2 + \| u \|_{L^2(\Omega)}^2 \rightarrow \min_{u > 0}.
\]
Weak formulation

\[
\frac{d\langle h, \varphi_{kl} \rangle}{dt} = -\langle u\partial_x h, \partial_x \varphi_{kl} \rangle - \langle u\partial_y h, \partial_y \varphi_{kl} \rangle + \langle W, \varphi_{kl} \rangle,
\]  \hspace{1cm} (11)

with initial condition \( \langle h(0, \cdot, \cdot) - h_0, \varphi_{kl} \rangle = 0 \) where \( \langle \cdot, \cdot \rangle \) denotes the inner product in \( L^2(0, a) \times L^2(0, b) \) and

\[
\varphi_{kl}(x, y) = \varphi_k(x)\varphi_l(y) = \sin\left(\frac{k\pi x}{a}\right)\cos\left(\frac{l\pi y}{b}\right),
\]  
k = 1 \ldots N_x, \ l = 0 \ldots N_y.
Weak formulation

\[
\frac{d\langle h, \varphi_{kl} \rangle}{dt} = -\langle u \partial_x h, \partial_x \varphi_{kl} \rangle - \langle u \partial_y h, \partial_y \varphi_{kl} \rangle + \langle W, \varphi_{kl} \rangle, \quad (11)
\]

with initial condition \(\langle h(0, \cdot, \cdot) - h_0, \varphi_{kl} \rangle = 0\) where \(\langle \cdot, \cdot \rangle\) denotes the inner product in \(L^2(0, a) \times L^2(0, b)\) and

\[
\varphi_{kl}(x, y) = \varphi_k(x) \varphi_l(y) = \sin\left(\frac{k\pi x}{a}\right) \cos\left(\frac{l\pi y}{b}\right),
\]

\(k = 1 \ldots N_x, \ l = 0 \ldots N_y\).

We further assume that the permeability field \(u\) is represented as follows:

\[
u(x, y) = \sum_{m,n=1}^{M_x,M_y} u_{mn} \psi^x_m(x) \psi^y_n(y) > 0, \quad (12)
\]

where \(\{\psi^x_m\}_{m \in \mathbb{N}}\) and \(\{\psi^y_n\}_{n \in \mathbb{N}}\) are total non-negative systems in \(L^2(0, a)\) and \(L^2(0, b)\) respectively.
Substituting the approximation \( h^N = \sum_{i=1,j=0}^{N_x,N_y} h_{ij} \varphi_{ij} \) into the weak formulation and taking into account the projection error we get:

\[
\frac{d\mathbf{h}}{dt} = -\sum_{m,n=1}^{M_x,M_y} u_{mn} A_{mn} \mathbf{h} + \mathbf{W}(t) + \mathbf{e}^m,
\]

\[0 = H_N \mathbf{h} + \mathbf{e}^o,
\]

\[h_{kl}(0) = \langle h_0, \varphi_{kl} \rangle, \quad k = 1, N_x, \quad l = 0, N_y,
\]

where \( \mathbf{h} = (h_1 \ldots h_{N_x,N_y})^T \) and \( \mathbf{W} = (W_{11} \ldots W_{N_xN_y})^T \) and

\[A_{mn} := \frac{4}{ab} \{ \langle \psi^x_m \psi^y_n \partial_x \varphi_{ij}, \partial_x \varphi_{kl} \rangle + \langle \psi^x_m \psi^y_n \partial_y \varphi_{ij}, \partial_y \varphi_{kl} \rangle \}^{N_x,N_y}_{k,i=1,l,j=0}.
\]
We rewrite the observation equation as follows:

\[ Y(t) = Ch + \eta, \]  

where \( Y = (Y_{11} \ldots Y_{p_xp_y})^T \), \( \eta \) absorbs the projection and observation errors and

\[ C = \left\{ \int_{\Omega} g_{kl}(x, y) \varphi_{ij}(x, y) \, dx \, dy \right\}^{p_x, p_y, N_x, N_y}_{k, l, i=1, j=0}. \]
Reduced control problem

\[ \sum_{s=1}^{M} \| R^{1/2} (Y(t_s) - Ch(t_s)) \|^2 + \sum_{m,n=1}^{M_x,M_y} (u_{mn})^2 \| \psi^x_m \psi^y_n \|_{L^2(\Omega)}^2 \rightarrow \min_{u_{mn}}, \]

\[ \frac{dh}{dt} = - \sum_{m,n=1}^{M_x,M_y} u_{mn}A_{mn}h + W + e^m, \]

\[ 0 = H_N h + e^o, \]

\[ h_{kl}(0) = \langle h_0, \varphi_{kl} \rangle, k = 1, N_x, l = 0, N_y. \]  \hspace{1cm} (15)

This problem is solved in two steps:

- optimization step: \( e^m, e^o \) are dropped and \( u_{mn} \) are approximated using Newton method;
Reduced control problem

\[
\sum_{s=1}^{M} \| R^{\frac{1}{2}} (Y(t_s) - Ch(t_s)) \|^2 + \sum_{m,n=1}^{M_x, M_y} (u_{mn})^2 \| \psi^x_m \psi^y_n \|^2_{L^2(\Omega)} \to \min_{u_{mn}},
\]

\[
dh \frac{dt}{dt} = - \sum_{m,n=1}^{M_x, M_y} u_{mn} A_{mn} h + W + e^m,
\]

\[
0 = H_N h + e^o,
\]

\[
h_{kl}(0) = \langle h_0, \varphi_{kl} \rangle, \ k = 1, N_x, \ l = 0, N_y.
\]  

(15)

This problem is solved in two steps:

- optimization step: \( e^m, e^o \) are dropped and \( u_{mn} \) are approximated using Newton method;
- filtering step: for the fixed \( u_{mn} \) the estimate of \( h \) is constructed.
Optimization problem

For the optimization problem:

\[
J(u) := \sum_{s=1}^{M} \| R^\frac{1}{2}(Y(t_s) - Ch(t_s)) \|^2 + \| \Psi^{\frac{1}{2}} u \|^2 \rightarrow \min_u ,
\]

\[
\frac{dh}{dt} = - \sum_{m,n=1}^{M_x,M_y} u_{mn} A_{mn} h + W , h(0) = h_0 ,
\]

the gradient and Jacobian are computed analytically:

\[
\nabla J(u) = 2 J^T(u)F(u) ,
\]

\[
F(u) = (R^{\frac{1}{2}}(Y(t_1) - Ch(t_1)))...R^{\frac{1}{2}}(Y(t_M) - Ch(t_M)) , \Psi^{\frac{1}{2}} u)^T ,
\]

\[
J(u) = \begin{pmatrix}
-CA_{11}(t_1 h(t_1) - z(t_1)) & ... & -CA_{M_x M_y}(t_1 h(t_1) - z(t_1)) \\
-CA_{11}(t_M h(t_M) - z(t_M)) & ... & -CA_{M_x M_y}(t_M h(t_M) - z(t_M))
\end{pmatrix},
\]

\[
\frac{dz}{dt} = - \sum_{m,n=1}^{M_x,M_y} u_{mn} A_{mn} z + tW(t) , z(0) = 0 .
\]
The Newton method reads as follows:

\[ u_{i+1} := u_i - \left( J^T(u_i)J(u_i) + \alpha I \right)^{-1} \nabla J(u_i), \quad u_0 = 0. \]

where \( \alpha > 0 \) is obtained through the line search:

\[ J(u_{i+1}(\alpha)) \rightarrow \min_{\alpha > 0} \]
Numerical experiment: setup

- 50x50 shifted Chebyshev polynomials to represent permeability $u$
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- 40x20 basis functions to represent $h$
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The problem is strongly ill-posed: given 30000 data points find 2500x800 parameters!
Numerical experiment: setup

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- 100 time-steps
Numerical experiment: setup

- 50x50 shifted Chebyshev polynomials to represent permeability $u$
- 40x20 basis functions to represent $h$
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- 100 time-steps
- the problem is strongly ill-posed: given 30000 data points find 2500x800 parameters!
Numerical experiment: true $u$
Numerical experiment: estimate
Numerical experiment

Truth

Estimate: relative error $\approx 25\%$
Numerical experiment: $L^2$-estimation error for $h$
Numerical experiment: summary

- 25% relative error in estimating $u$
- Less than 2% relative error in estimating $h$
- $J(\hat{u}) \approx 0.06$
- Relative error in estimating the state transition matrix $\sum_{mn} A_{mn}$ is $\approx 13$
- Multiple global minima?
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• multiple global minima?
Conclusions

• robust extension of Galerkin projection method allows to model the projection coefficients in the closed form and produce worst-case estimation error estimate;
• method was applied to:
  • filtering problem for linear transport equation with strongly occluded observations (Zhuk, Frank, Herlin, Shorten, 2013, submitted);
  • inversion problem for 2D linear Darcy equation with heterogeneous diffusion coefficient (Zhuk, McKenna, 2013, in preparation).