Preprints of the 16th IFAC Symposium on System Identification
The International Federation of Automatic Control
Brussels, Belgium. July 11-13, 2012

Minimax state estimation for linear stationary differential-algebraic equations ⋆

Sergiy Zhuk *

* *IBM Research, Dublin, Ireland (Tel: 35 31 826 94 51; e-mail: sergiy.zhuk@ie.ibm.com).*

Abstract: This paper presents a generalization of the minimax state estimation approach for singular linear Differential-Algebraic Equations (DAE) with uncertain but bounded input and observation's noise. We apply generalized Kalman Duality principle to DAE in order to represent the minimax estimate as a solution of a dual control problem for adjoint DAE. The latter is then solved converting the adjoint DAE into ODE by means of a projection algorithm. Finally, we represent the minimax estimate in the form of a linear recursive filter.

Keywords: Minimax; Robust estimation; Descriptor systems; Optimization under uncertainties.

1. INTRODUCTION

This paper presents a generalization of the minimax state estimation approach to linear Differential-Algebraic Equation (DAE) in the form

$$\frac{d(Fx)}{dt} = Ax(t) + f(t), \quad Fx(t_0) = x_0\,, \tag{1}$$

where $F, A \in \mathbb{R}^{m\times n}$. State estimation theory is one of the main tools in mathematical modeling. It is used, in particular, for parameter identification in oceanography [Heitz et al., 2010] or operational forecasts in meteorology and air pollution [Wu et al., 2008]. Mathematically, geophysical models are represented by systems of Partial Differential Equations (PDE) that make state estimation for those models very expensive from the computational point of view. In practice, different model reduction techniques are applied in order to get a reasonable computational time. For instance, the classical Galerkin projection method allows one to project the state of an infinite dimensional state equation, described by a PDE, onto a finite dimensional manifold and focus on the dynamics of the projection coefficients only. In the closed form this dynamics may be represented by the DAE (1): the main idea is to model the impact of the unresolved part of the state vector onto projection coefficients introducing the model error f and restricting it to belong to a certain subspace [Mallet and Zhuk, 2011]. Apart from the model reduction DAEs are applied in robotics [Schiehlen, 2000]. Pros and cons of using DAEs for modeling were discussed by Müller [2000].

The common way of deriving a state estimate for DAE is to 1) convert the DAE into an Ordinary Differential Equation (ODE) transforming the matrix pencil $F - \lambda A$ to Weierstrass canonical form [Gantmacher, 1960] and 2) apply the classical Kalman Duality (KD) principle to the resulting ODE. For the detailed description of 1), we refer the reader to [Darouach et al., 1997], [Gerdin et al., 2007] and citations there. We note that applying the matrix pencils theory to DAEs one restricts coefficients F and A (det $(F - \lambda A) \neq 0$ for some λ) and might need to differentiate f. Although smoothness of f may be appropriate for control problems, it turns out to be a very restrictive assumption in the context of state estimation where f is the model error which is often represented by a measurable function with bounded $\mathbb{L}_2$-norm. In what follows we derive a minimax state estimate for DAE (1) without these limitations.

The main contribution of this paper is an optimal recursive state estimation algorithm for stationary DAEs in the form (1). The algorithm is applied to DAE directly that allows to avoid the restrictions imposed by the matrix pencils theory. To achieve this we transform the state estimation problem for DAE (1) to the dual control problem by means of generalized Kalman Duality (KD) principle proposed by Zhuk [2012] and the latter is then solved using a projection method. For the case of ellipsoidal $\mathscr{G}$ (see formula (7)) generalized KD states that the optimal estimate $\hat{u}(y) = \int_{t_0}^{t_1} \hat{u}^T y dt$ of the linear function $\ell^T Fx(t_1)$ may be found as a solution of quadratic control problem for adjoint DAE:

$$\frac{d(F'z)}{dt} = -A'z(t) + H'u(t), \quad F'z(t_1) = F'\ell\,. \tag{2}$$

If $F = I$ then this statement reduces to the classical KD principle (see [Åström, 2006]). In the general case, the set of all $\ell \in \mathbb{R}^m$ such that (2) has a solution — this set will be referred to as a minimax observable subspace $\mathscr{L}(t_1)$ — describes all possible directions in the state space of (1) that have the following property: if $\ell \in \mathscr{L}(t_1)$ then the worst-case estimation error (see formula (9)) is finite. In other words, the estimate $\hat{u}(y)$ with finite worst-case estimation error may be constructed for the projection of $Fx(t_1)$ on $\mathscr{L}(t_1)$ only. It is worth noting that for all $\ell \in \mathscr{L}(t_1)$ the estimate of $\ell^F x(t_1)$ corresponds, in fact, to the solution of (1) which has the smallest eucledian norm and is, therefore, unique (see formula (5) and the following discussion). Therefore, there is no need to restrict F and A in order to force unique solvability for (1). Instead, it

⋆ This work was carried out during the author's tenure of an ERCIM "Alain Bensoussan" Fellowship Programme. This Programme is supported by the Marie Curie Co-funding of Regional, National and International Programmes (COFUND) of the European Commission.

is sufficient to take ℓ from $\mathscr{L}(t_1)$ and so constraints are moved from F, A onto ℓ.

We present a projection algorithm allowing one to convert (2) into equivalent ODE. It efficiently computes the minimax observable subspace and determines algebraic constraints for u imposed by the structure of (2). The algorithm splits DAE (2) into differential and algebraic parts (see (13)-(14)) and constructs an ODE (see (22)) such that all its solutions belong to the linear manifold defined by the algebraic part of DAE. This ODE is obtained applying a finite number of linear transformations (see (15)- (18)) to the differential part of DAE. We stress that in the general case the number of required transformations would be infinite if the DAE was non-stationary. An alternative way of constructing the set of all solutions for (2) is based on the theory of singular matrix pencils. We stress that this approach constrains u and its derivatives as well (see [Gantmacher, 1960, p.336,form.(79)]) that, in turn, complicates the procedure of computing the optimal control.

Transforming the cost function of the dual control problem we find an optimal estimate $\hat{u}$ using Pontryagin maximum principle. We note that necessary optimality conditions for linear quadratic control problem with stationary regular DAE as a constraint were derived in Bender and Laub [1987]. The general case of a singular non-stationary DAE was considered by Zhuk [2012] where Tikhonov regularization is used to derive optimality conditions. Although the approach of this paper is applicable to the class of DAEs with constant coefficients only, it gives exact necessary and sufficient optimality conditions in contrast to the general approach presented in [Zhuk, 2012].

Finally, we derive the minimax estimate in the form of the linear recursive filter. We refer the reader to [Milanese and Tempo, 1985], [Chernousko, 1994] and [Kurzhanski and Vályi, 1997] for the basic information on the minimax framework. Minimax estimates for linear singular DAEs with discrete time were obtained by Zhuk [2010]. Let us note that our approach is applicable for the case of unbounded input f: indeed, one can always introduce an extended state $\tilde{x} = (x, f)'$, a new matrix $F_1 = \begin{pmatrix} F & 0 \end{pmatrix}$ and add a dummy bounded input f_1 in (1). In this regard let us mention H_∞ framework [Başar and Bernhard, 1995] which allows one to deal with unbounded f. Connections between minimax and H_∞ frameworks were revealed in [Baras and Kurzhanski, 1995].

This paper is organized as follows. At the beginning of section 2 we describe the formal problem statement and definitions, and discuss ill-posedness of DAE. Then we present the dual control problem (Proposition 2) and projection algorithm (Proposition 3). At the end of the section the state estimation algorithm is derived (Theorem 4). In section 3 we consider a simple ODE with unbounded inputs and describe its minimax observable subspace and derive the minimax estimate. Section 4 contains conclusions. All proofs and technical statements are located in Appendix.

Notation: $\mathbb{R}^n$ denotes the n-dimensional Euclidean space; $\mathbb{L}_2(t_0, t_1, \mathbb{R}^m)$ denotes a space of square-integrable functions with values in $\mathbb{R}^m$ (in what follows we will often write $\mathbb{L}_2(t_0, t_1)$ referring $\mathbb{L}_2(t_0, t_1, \mathbb{R}^k)$ where the dimension k will be defined by the context); $\mathbb{H}_1(t_0, t_1, \mathbb{R}^m)$ denotes a space of absolutely continuous functions with $\mathbb{L}_2(t_0, t_1)$-derivative; the prime $'$ denotes the operation of taking the adjoint: L' denotes adjoint operator, F' denotes the transposed matrix; $\mathscr{R}(L)$, $\mathscr{N}(L)$ denote the range and null-space of the operator L, $P_R(F)(P_N(F))$ denotes the orthogonal projection matrix onto the range (null-space) of the matrix F; $c(G, \cdot)$ denotes the support function of a set G; $\delta(G, \cdot)$ denotes the indicator of G: $\delta(G, f) = 0$ if $f \in G$ and $+\infty$ otherwise; $x^T y$ denotes the inner product of vectors $x, y \in \mathbb{R}^n$, $\|x\|^2 := x^T x$; $S > 0$ means $x^T S x > 0$ for all $x \in \mathbb{R}^n$; F^+ denotes the pseudoinverse matrix; I_n denotes $n \times n$-identity matrix, $0_{n\times m}$ denotes $n \times m$-zero matrix, $I_0 := 0$;

2. LINEAR MINIMAX ESTIMATION FOR DAES

Consider a pair of systems

$$\frac{d(Fx)}{dt} = Ax(t) + f(t), \quad Fx(t_0) = x_0, \tag{3}$$

$$y(t) = Hx(t) + \eta(t), \tag{4}$$

where $F, A \in \mathbb{R}^{m\times n}$, $H \in \mathbb{R}^{p\times n}$, $t_0, t_1 \in \mathbb{R}$ and $x(t) \in \mathbb{R}^n$, $f(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$, $\eta(t) \in \mathbb{R}^p$ represent the state, input, observation and observation's noise respectively.

In what follows we assume that initial condition x_0, input f and observation's noise η are unknown and belong to the given bounding set $\mathscr{G}$. Our aim is to estimate the state $Fx(t_1)$ of (3) given observations $y(t)$, $t \in (t_0, t_1)$. Let us briefly describe the main points of our strategy. In the case $F = I$ the classical state estimation procedure (see [Åström, 2006]) suggests to look for an estimate of a linear function $\ell(x) := \ell^T F x(t_1)$ in the class of linear functions $u(y) := \int_{t_0}^{t_1} u^T(t) y(t) dt$, provided x solves (3) and the output y is given in the form (4). One computes an estimate $\hat{u}$ which is optimal with respect to some criteria $\tilde{\sigma}$. Assuming that $\mathscr{G}$ is bounded, $\tilde{\sigma}$ may be defined (see [Nakonechny, 1978] for details) as a worst-case estimation error

$$\tilde{\sigma}(t_1, \ell, u) := \sup_{(x_0, f, \eta)\in\mathscr{G}} (\ell(x) - u(y))^2$$

where x is a unique solution of ODE (3) corresponding to data x_0, f. In this case, the optimal estimate $\hat{u}$ solves the equation $\tilde{\sigma}(t_1, \ell, \hat{u}) = \inf_u \tilde{\sigma}(\ell, u)$. In general case $F \in \mathbb{R}^{m\times n}$ (3) is solvable not for all x_0 and f, and so one needs to assume that there is at least one x_0, f, η in $\mathscr{G}$ such that (3) has a solution. Further, if (3) is solvable for a given x_0, f, it may have non-unique solution so $\tilde{\sigma}(t_1, \ell, u)$ is not well-defined. In fact, any solution of (3) corresponding to the data x_0, f admits the following representation:

$$x = x^0 + x^1, \quad x^0 \in \mathscr{X}_0, x^1 \perp \mathscr{X}_0, \tag{5}$$

where $\mathscr{X}_0$ contains all x such that x solves (3) with $x_0 = 0$ and $f = 0$. We stress that the data x_0, f determines x^1 uniquely and so we may define the worst-case error as follows:

$$\sigma(t_1, \ell, u) := \sup_{x\in\mathscr{X}_0} \sup_{x_0, f, \eta} (\ell(x) - u(y))^2,$$

where an additional sup is taken over the set $\mathscr{X}_0$. This allows one to eliminate the part of the solution x belonging to $\mathscr{X}_0$. Notice that, in contrast with ODEs, $\sigma(t_1, \ell, u)$ may be infinite for some ℓ, u as $\mathscr{X}_0$ is a linear subspace. This observation suggests to introduce a so called minimax observable subspace $\mathscr{L}(t_1)$.

Definition 1. Given $t_1 < +\infty$, $u \in \mathbb{L}_2(t_0, t_1, \mathbb{R}^p)$ and $\ell \in \mathbb{R}^m$ define a worst-case estimation error

$$\sigma(t_1, \ell, u) := \sup_{x \in \mathscr{X}_0, (x_0, f, \eta) \in \mathscr{G}} (\ell^T F x(t_1) - \int_{t_0}^{t_1} u^T(t) y(t) dt)^2$$

A function $\hat{u}(y) = \int_{t_0}^{t_1} \hat{u}^T(t) y(t) dt$ is called a minimax estimate in the direction ℓ (ℓ-estimate) if $\inf_u \sigma(t_1, \ell, u) = \sigma(t_1, \ell, \hat{u})$. The number $\hat{\sigma}(t_1, \ell) := \sigma(t_1, \ell, \hat{u})$ is called a minimax error in the direction ℓ at time-instant t_1 (ℓ-error). The set $\mathscr{L}(t_1) := \{\ell \in \mathbb{R}^m : \hat{\sigma}(t_1, \ell) < +\infty\}$ is called a minimax observable subspace.

We impose the following uncertainty description. Let $Q_0, Q(t) \in \mathbb{R}^{m \times m}$, $Q_0 = Q_0' > 0$, $Q = Q' > 0$, $R(t) \in \mathbb{R}^{p \times p}$, $R' = R > 0$; $Q(t)$, $R(t)$, $R^{-1}(t)$ and $Q^{-1}(t)$ are continuous functions of t on $[t_0, t_1]$. Define

$$E(x_0, f, \eta) := x_0^T Q_0 x_0 + \int_{t_0}^{t_1} f^T Q(t) f + \eta^T R(t) \eta dt \quad (6)$$

and let

$$\mathscr{G} = \{(x_0, f, \eta) : E(x_0, f, \eta) \le 1\} \quad (7)$$

It is easy to see that there is at least one $(x_0, f, \eta) \in \mathscr{G}$ such that (3) has a solution x.

A geometrical representation of the ℓ-estimate and error. Define a linear function $\mathcal{F}_u(x, \eta) := \ell^T F x(t_1) - \int_{t_0}^{t_1} u^T(t) y(t) dt$ and operator

$$(Lx)(t) = \left(Fx(t_0), \frac{d(Fx)}{dt} - Ax(t)\right), \; x \in \mathscr{D}(L)$$
$$\mathscr{D}(L) := \{x \in \mathbb{L}_2(t_0, t_1) : Fx \in \mathbb{H}_1(t_0, t_1, \mathbb{R}^m)\} \quad (8)$$

and note that (3) is equivalent to $Lx(t) = (x_0, f(t))$ and $\mathscr{X}_0 = \mathscr{N}(L)$. Since $(x_0, f, \eta) \in \mathscr{G}$ we can write that:

$$\sigma^{\frac{1}{2}}(t_1, \ell, u) = \sup_{x \in \mathscr{D}(L), \eta} \{\mathcal{F}_u(x, \eta) : (Lx, \eta) \in \mathscr{G}\} \quad (9)$$

so that ℓ-error is a support function of the convex set

$$\mathscr{X} = \{(x, \eta) : (Lx, \eta) \in \mathscr{G}\} \quad (10)$$

The support function of $\mathscr{X}$ describes the distance of the supporting hyperplane $\mathscr{H}(u) := \{(x, \eta) : \mathcal{F}_u(x, \eta) \le \sigma^{\frac{1}{2}}(t_1, \ell, u)\}$ from the origin. Then, the ℓ-estimate $\hat{u}$ defines a direction in a functional space $\mathbb{L}_2(t_0, t_1)$ such that the distance of $\mathscr{H}(\hat{u})$ from the origin is minimal and so in this direction the maximal deviation of elements of $\mathscr{X}$ from the origin is minimal.

Proposition 2. (dual control problem). Take $\ell \in \mathbb{R}^m$. The ℓ-error $\hat{\sigma}(t_1, \ell)$ is finite iff there are $z \in \mathbb{L}_2(0, t_1, \mathbb{R}^m)$ and $u \in \mathbb{L}_2(0, t_1, \mathbb{R}^p)$ such that:

$$\frac{d(F'z)}{dt} = -A'z(t) + H'u(t), \quad F'z(t_1) = F'\ell. \quad (11)$$

If $\hat{\sigma}(t_1, \ell) < +\infty$ then ℓ-estimate $\hat{u}$ is a unique solution of the following optimal control problem:

$$\mathscr{I}(z, d, u) := \int_{t_0}^{t_1} (u^T R^{-1} u + z^T Q^{-1} z) dt (:= \mathscr{I}_1(z, u))$$
$$+ (F'^+ F' z(t_0) - d)^T Q_0^{-1} (F'^+ F' z(t_0) - d) \to \min_{z, d, u}$$
$$\frac{d(F'z)}{dt} = -A'z + H'u, \quad F'd = 0, \; F'z(t_1) = F'\ell. \quad (12)$$

Next proposition presents the projection algorithm allowing to convert DAE to ODE.

Proposition 3. Take $A_{1,2}^0 \in \mathbb{R}^{n \times n}$, $A_{3,4}^0 \in \mathbb{R}^{m \times n}$ and consider DAE

$$\frac{dp}{dt} = A_1^0 p + A_2^0 q, \quad p(t_1) = w, \quad (13)$$
$$0 = A_3^0 p + A_4^0 q. \quad (14)$$

Let us set by definition

$$A_1^{k+1} = (A_1^k - A_2^k (A_4^k)^+ A_3^k) P_N\big[(I - P_R(A_4^k)) A_3^k\big], \quad (15)$$
$$A_2^{k+1} = A_2^k P_N(A_4^k), \quad (16)$$
$$A_3^{k+1} = (I - P_N\big[(I - P_R(A_4^k)) A_3^k\big]) A_1^{k+1}, \quad (17)$$
$$A_4^{k+1} = (I - P_N\big[(I - P_R(A_4^k)) A_3^k\big]) A_2^{k+1}, \quad (18)$$
$$P_i^s := \sqcap_{k=i}^s P_N\big[(I - P_R(A_4^k)) A_3^k\big], \quad (19)$$
$$p^k = P_N\big[(I - P_R(A_4^k)) A_3^k\big] p^{k+1}, \; p^j = p^*, \quad (20)$$
$$q^k = -(A_4^k)^+ A_3^k p^k + P_N(A_4^k) q^{k+1}, \; q^j = q^*. \quad (21)$$

Then 1) there exists $s \le n$ such that $A_3^s = A_4^s = 0_{m \times n}$; 2) DAE (13)-(14) has a solution p, q iff $w = P_0^s w$; 3) if p, q solve (13)-(14) then $p = p^0$, $q = q^0$ where p^0, q^0 are defined from (20)-(21) with $p^* := \overline{p}$, $q^* := \overline{q}$, $k \in \overline{0, s-1}$ and

$$\frac{d\overline{p}}{dt} = A_1^s \overline{p}(t) + A_2^s \overline{q}(t), \quad \overline{p}(t_1) = w, \overline{q} \in \mathbb{L}_2(t_0, t_1). \quad (22)$$

Let us define $\tilde{I} = \begin{pmatrix} I_m \\ 0_{p \times m} \end{pmatrix}$ and set:

$$A_1^0 = -(F^+)' A' P_R(F), \; A_2^0 = \begin{pmatrix} -(F^+)' A' P_N(F') & (F^+)' H' \end{pmatrix},$$
$$A_3^0 = -P_N(F) A' P_R(F), \; A_4^0 = \begin{pmatrix} -P_N(F) A' P_N(F') & P_N(F) H' \end{pmatrix}$$
$$\tilde{Q} = \begin{pmatrix} Q^{-1} & 0 \\ 0 & R^{-1} \end{pmatrix}, \; \beta^s := \begin{pmatrix} P_N(F') & 0 \\ 0 & I_p \end{pmatrix} \times \sqcap_{i=0}^{s-1} P_N(A_4^i),$$
$$\gamma := \beta^{s\prime} \tilde{Q} \beta^s, \; \alpha^s := -\sum_{i=0}^{s-1} P_N(A_4^{i-1}) (A_4^i)^+ A_3^i P_i^s, \; A_4^{-1} = 0,$$
$$D := (I - (P_N(F') Q_0^{-1} P_N(F'))^+ P_N(F') Q_0^{-1}),$$
$$\overline{Q} := \alpha^{s\prime} \tilde{Q} \alpha^s + P_0^{s\prime} \tilde{I}' \tilde{Q} \tilde{I} P_0^s + \alpha^{s\prime} \tilde{Q} \tilde{I} P_0^s + P_0^{s\prime} \tilde{I}' \tilde{Q} \alpha^s,$$
$$B^s := \alpha^s + \tilde{I} P_0^s, \; \tilde{Q}_0 := D' Q_0^{-1} D, \; \overline{Q}_0 := P_0^{s\prime} \tilde{Q}_0 P_0^s.$$

The next theorem describes the minimax observable subspace, ℓ-error and represents ℓ-estimate in the form of a linear recursive filter on a subspace.

Theorem 4. Assume that $s \le n$ is defined as in Proposition 3 and define

$$\frac{dK}{dt} = (-A_1^{s\prime} + B^{s\prime} \tilde{Q} \beta^s \gamma^+ A_2^{s\prime}) K$$
$$+ K(-A_1^s + A_2^s \gamma^+ \beta^{s\prime} \tilde{Q} B^s) - K A_2^s \gamma^+ A_2^{s\prime} K$$
$$+ \overline{Q} - B^{s\prime} \tilde{Q} \beta^s \gamma^+ \beta^{s\prime} \tilde{Q} B^s, \; K(t_0) = \overline{Q}_0^+,$$
$$\frac{d\hat{x}}{dt} = (-A_1^{s\prime} + B^{s\prime} \tilde{Q} \beta^s \gamma^+ A_2^{s\prime} - K A_2^s \gamma^+ A_2^{s\prime}) \hat{x}$$
$$+ (\alpha^{s\prime} + (K A_2^s - B^{s\prime} \tilde{Q} \beta^s) \gamma^+ \beta^{s\prime}) \begin{pmatrix} 0 \\ y \end{pmatrix}, \; \hat{x}(t_0) = 0.$$

Then $\hat{u}(y) = \ell^T F F^+ \hat{x}(t_1)$ and

$$\hat{\sigma}(t_1, \ell) = \ell^T F F^+ K(t_1) F F^+ \ell, \; \forall \ell \in \mathscr{L}(t_1),$$
$$\mathscr{L}(t_1) = \{\ell \in \mathbb{R}^n : P_0^s (F^+)' F' \ell = (F^+)' F' \ell\}. \quad (23)$$

3. EXAMPLE

This example illustrates application of Theorem 4 to an ill-posed DAE which represents an ODE with unbounded inputs. As it was pointed out in Zhuk [2012] application of Pontryagin maximum principle to the dual control problem for this DAE would lead to the restriction of the minimax observable subspace. We apply a projection algorithm

(Proposition 3) in order to preserve the structure of $\mathscr{L}(t_1)$.
Let $F = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix}$, $H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$.
Then (3)-(4) has the following form:($\dot{x} := \frac{dx}{dt}$):

$$\dot{x}_1 = x_3 + f_1,\ \dot{x}_2 = -x_1 - x_4 + f_2, \quad x_{1,2}(t_0) = x_0^{1,2},$$
$$y_1 = x_1 + \eta_1,\ y_2 = x_4 + \eta_2,\ y_3 = x_2 + \eta_3\,.$$

where $x_0^{1,2}$, $f_{1,2}$ and $\eta_{1,2,3}$ belong to (7) and E in (6) is defined with $Q_0 = I_2$, $Q = I_2$, $R = I_3$. We see that $x_{3,4} \in \mathbb{L}_2(t_0, t_1)$ are arbitrary functions. In this case, by Proposition 2 the ℓ-estimate $\hat{u}$ of $Fx(t_1)$ may be obtained minimizing the worst-case error

$$\sigma(t_1, \ell, u) = \sum_{i=1}^{2} \|z_i(t_0)\|^2 + \int_{t_0}^{t_1} z_i^2 dt + \int_{t_0}^{t_1} \sum_{j=1}^{3} u_j^2 dt \quad (24)$$

over solutions to the adjoint DAE:

$$\begin{aligned} &\dot{z}_1 - z_2 - u_1 = 0, \quad z_1(t_1) = \ell_1, \quad -z_1 \equiv 0\,, \\ &\dot{z}_2 - u_3 = 0, \quad z_2(t_1) = \ell_2, \quad -z_2 - u_2 = 0\,. \end{aligned} \quad (25)$$

The structure of (25) is simple and so we can explicitly compute $\hat{u}$ minimizing (25). This will allow to compare the corresponding ℓ-estimate with the one derived from Theorem 4. (25) implies that $\hat{u}_{1,2} = -z_2$. Hence, $\hat{u}_3$ is a unique solution to the following control problem

$$\sigma(t_1, \ell, u) = \|z_2(t_0)\|^2 + \int_{t_0}^{t_1} 3z_2^2 + u_3^2 dt,$$
$$\dot{z}_2 = u_3, \quad z_2(t_1) = \ell_2\,.$$

The optimality condition takes the classical form: $\hat{u}_3 = p$ where

$$\dot{z}_2 = p, \quad z_2(t_1) = \ell_2,$$
$$\dot{p} = 3z_2, \quad p(t_0) = z_2(t_0).$$

Introducing k as a solution of the Riccati equation $\dot{k} = 3 - k^2$, $k(0) = 1$ we find that $\hat{u}_3 = kz_2$ where $\dot{z}_2 = kz_2$, $z_2(t_1) = \ell_2$. Let $\hat{x}$ be a solution to

$$\dot{\hat{x}} = -k\hat{x} - y_1 - y_2 + ky_3, \quad \hat{x}(t_0) = 0\,. \quad (26)$$

Then it is easy to see that $\hat{u}(y) = \int_{t_0}^{t_1} \hat{u}^T y dt = \ell_2 \hat{x}(t_1)$ and so $\hat{x}$ represents a minimax state estimate.
Now, let us use Proposition 3. We find that $s = 2$, $\mathscr{L}(t_1) = \{0\} \times R$ and

$$\begin{aligned} &\begin{pmatrix} \dot{k}_1 & \dot{k}_2 \\ \dot{k}_2 & \dot{k}_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} - \begin{pmatrix} k_1 & k_2 \\ k_2 & k_4 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} k_1 & k_2 \\ k_2 & k_4 \end{pmatrix}, \\ &\dot{\hat{x}}_1 = k_2 y_3, \quad \dot{\hat{x}}_2 = -k_2\hat{x}_1 - k_4\hat{x}_2 - y_1 - y_2 + k_4 y_3, \end{aligned} \quad (27)$$

with $\hat{x}_{1,2}(t_0) = 0$ and $\begin{pmatrix} k_1(t_0) & k_2(t_0) \\ k_2(t_0) & k_4(t_0) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. From the first equation of (27) we see that $k_1 = k_2 = 0$ so that $\hat{x}_1 = 0$ and the equation for $\hat{x}_2$ coincides with (26).
In order to generate $y_{1,2,3}$ we take $t_0 = 0$, $t_1 = 1$, $x_3 = \cos(t)$ and $x_4 = \sin(t)$, $x_1(0) = 0.1$, $x_2(0) = -0.1$, $f_1 = f_2 = 0$, $\eta_1 = -0.1$, $\eta_2 = -0.2$, $\eta_3 = 0.3$. In Figure 1 the ℓ-estimate $\hat{u}(y)$, and ℓ-error are presented. As $\mathcal{L}(t) \equiv \{0\} \times \mathbb{R}$ we see that x_1 is not observable in the minimax sense although $y_1 = x_1 + \eta_1$ is observed. This can be explained as follows: the derivative x_3 of x_1 may be any element of $\mathbb{L}_2$. It is not hard to compute that the expression for $\sigma(t_1, \ell, u)$ contains $\int_{t_0}^{T} x_3 z_1 dt$ and so $\sigma(t_1, \ell, u) < +\infty$ if and only if $z_1(t) \equiv 0$ for any $t \in (t_0, t_1)$. As z_1 is absolutely continuous it follows that $z_1(t_1) = 0$. Hence, if $\ell_1 = 0$ then ℓ-error is finite. If $\ell_1 \neq 0$ then the only candidate for $\hat{u}_1$ is the impulse

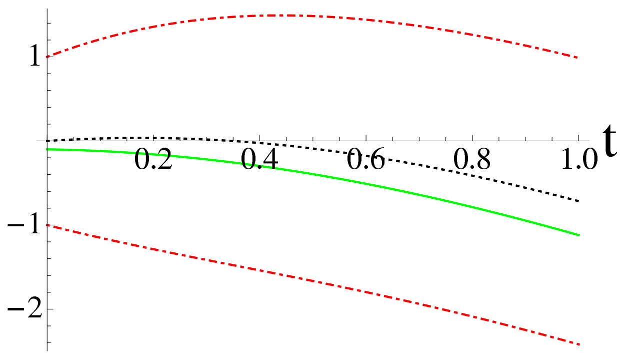

Fig. 1. ℓ-estimate $\hat{u}_\varepsilon(y) = \hat{x}_2(t)$ (Dotted), error $\hat{\sigma}(t, \ell)$ (DotDashed) and simulated trajectory $x_2(t)$ (Solid), $t \in [0, 1]$. It is easy to see that the minimax estimate is a "central" point of the set of all possible realizations of the "true" state x_2. The estimate is "biased" due to the presence of unknown functions x_2, x_3. Nevertheless, the minimax error stays bounded at infinity that proves robustness of the ℓ-estimate with respect to unknown and possibly unbounded disturbances in the state equation.

control $\delta(t_1 - t)\ell_1$ switching z_1 from 0 to ℓ_1 at time-instant t_1. However, $\delta \notin \mathbb{L}_2$ so that ℓ-error is infinite. In other words, the algebraic structure of the adjoint DAE (25) can not "compensate" the unbounded derivative of x_1 though it compensates the unbounded derivative of x_2 and so the minimax observable subspace is non-trivial in that direction. It is curious to note that minimax error in the direction $(0, 1)$ stays bounded with time as the solution of Riccati equation is bounded at infinity.

4. CONCLUSION

In this paper we discuss the minimax state estimation approach for linear stationary DAEs with unknown but bounded initial condition, input and observation noise. Our approach is based on the generalized Kalman duality principle that allows to avoid constraints on DAE coefficients and input which are usually imposed in order to apply the theory of matrix pencils. The presented approach is relevant to high-dimensional numerical models that require state estimation algorithms in reduced form due to computational burden. In this case F might encapsulate the vectors which span a low dimensional reduced sub-space and the state of the resulting DAE describes the evolution of the projection coefficients. This reduction approach addresses stability issues of the classical Galerkin projection method (see [Mallet and Zhuk, 2010] for details). In perspective, we plan to apply the presented duality concept to stochastic DAEs. Another promising direction is an ensemble state estimation where one state is shared by different models and each model describes the dynamics of the same process reflecting various physical/chemical/numerical parameterizations for the same process (see [Garaud and Mallet, 2011] for an example in air quality modeling).

REFERENCES

A. Albert. *Regression and the Moor-Penrose pseudoinverse*. Acad. press, N.Y., 1972.

K. Åström. *Introduction to Stochastic Control*. Dover, 2006.

T. Başar and P. Bernhard. *H_∞-optimal Control and Related Minimax Design Problems*. Springer, 1995.

J. S. Baras and A.B. Kurzhanski. Nonlinear filtering: The set-membership and the H_∞ techniques. In *Proc. 3rd IFAC Symp.Nonlinear Control Sys.Design*. Pergamon, 1995.

D. Bender and A. Laub. The linear-quadratic optimal regulator for descriptor systems. *IEEE TAC*, 32(8):672–688, 1987.

F. L. Chernousko. *State Estimation for Dynamic Systems* . Boca Raton, FL: CRC, 1994.

M. Darouach, M. Boutayeb, and M. Zasadzinski. Kalman filtering for continuous descriptor systems. In *ACC*, pages 2108–2112, Albuquerque, June 1997. AACC.

W. Fleming and R. Rishel. *Deterministic and Stochastic Optimal control*. Springer, 1975.

F. Gantmacher. *The theory of matrices*. Chelsea Publish.Comp., N.-Y., 1960.

Damien Garaud and Vivien Mallet. Automatic calibration of an ensemble for uncertainty estimation and probabilistic forecast: Application to air quality. *J. Geophys. Res.*, 116, 2011.

Markus Gerdin, Thomas B. Schön, Torkel Glad, Fredrik Gustafsson, and Lennart Ljung. On parameter and state estimation for linear differential-algebraic equations. *Automatica J. IFAC*, 43(3):416–425, 2007.

D. Heitz, E. Mémin, and C. Schnörr. Variational fluid flow measurements from image sequences: synopsis and perspectives. *Exp. Fluids*, 48(3):369–393, 2010.

A. Ioffe and V. Tikhomirov. *Theory of extremal problems*. North-Holland, Amsterdam, 1974.

Alexander Kurzhanski and István Vályi. *Ellipsoidal calculus for estimation and control*. Systems & Control: Foundations & Applications. Birkhäuser Boston Inc., Boston, MA, 1997. ISBN 0-8176-3699-4.

V. Mallet and S. Zhuk. Reduced minimax state estimation. Technical Report RR-7500, INRIA, Paris-Rocquencourt, 2010. `http://hal.archives-ouvertes.fr`.

V. Mallet and S. Zhuk. Reduced minimax filtering by means of differential-algebraic equations. In *Proc. of 5th Int. Conf. on Physics and Control (PhysCon 2011)*. IPACS Electronic Library, 2011. Available at: `lib.physcon.ru`.

Mario Milanese and Roberto Tempo. Optimal algorithms theory for robust estimation and prediction. *IEEE Trans. Automat. Control*, 30(8):730–738, 1985.

P. Müller. Descriptor systems: pros and cons of system modelling by differential-algebraic equations. *Mathematics and Computers in Simulation*, 53(4-6):273–279, 2000.

A. Nakonechny. A minimax estimate for functionals of the solutions of operator equations. *Arch. Math. (Brno)*, 14 (1):55–59, 1978. ISSN 0044-8753.

W. Schiehlen. Force coupling versus differential algebraic description of constrained multibody systems. *Multibody System Dynamics*, 4:317–340, 2000.

Lin Wu, Vivien Mallet, Marc Bocquet, and Bruno Sportisse. A comparison study of data assimilation algorithms for ozone forecasts. *J. Geophys. Res.*, 113 (D20310), 2008.

S. Zhuk. Minimax state estimation for linear discrete-time differential-algebraic equations. *Automatica J. IFAC*, 46(11):1785–1789, 2010. doi: 10.1016/j.automatica.2010.06.040.

S. Zhuk. Kalman duality principle for a class of ill-posed minimax control problems with linear differential-algebraic constraints. *Applied Mathematics & Optimisation*, 2012. submitted.

APPENDIX

Proof of Proposition 2. Generalized Kalman duality for DAEs in the form (3) with zero initial condition was presented in [Zhuk, 2012, Th.2.4]. The proof given there is based on the operator interpretation of DAEs (see geometrical representation of ℓ-estimate above). Applying the method of Zhuk [2012] (see the first part of the proof of Theorem 2.4) to the definition of operator L given by formula (8) it is easy to see that (11) has a solution and (see [Zhuk, 2012, frml.(2.10)]):

$$\sigma^{\frac{1}{2}}(t_1,\ell,u) = \min_{(z_0,v)\in\mathscr{N}(L')} c(\mathscr{G}, F'^{+}F'z(t_0)-z_0, z-v, u)\,, \tag{.1}$$

where L' is defined as follows:

$$\begin{aligned}&(L'b)(t) = -\frac{dF'v}{dt} - A'v, b\in\mathscr{D}(L'),\\ &\mathscr{D}(L') := \{b=(z_0,v) : F'v\in\mathbb{H}_1(t_0,t_1,\mathbb{R}^m),\\ &F'v(t_1)=0, v_0 = F^{+'}F'v(t_0)+d, F'd=0\}\,.\end{aligned} \tag{.2}$$

Since $(z_0,v)\in\mathscr{N}(L')$ and z solves (11), it follows that $z+v$ solves (11), and so taking the min in (.1) with respect to $(z_0,v)\in\mathscr{N}(L')$ is equivalent to minimize with respect to all possible d and z such that $d: F'd=0$ and z solves (11) for the fixed u:

$$\sigma^{\frac{1}{2}}(t_1,\ell,u) = \min_{z \text{ solves (11) and } F'd=0} c(\mathscr{G}, F'^{+}F'z(t_0)-d, z, u) \tag{.3}$$

and min is attained at $\tilde z, \tilde d$. Noting that the support function of the ellipsoid $\mathscr{G}$ coincides with $\mathscr{I}$, namely $c^2(\mathscr{G}, F'^{+}F'z(t_0)-d,z,u) = \mathscr{I}(z,d,u)$ and recalling (.3) we obtain

$$\sigma(t_1,\ell,u) = \min_{z \text{ solves (11) and } F'd=0} \mathscr{I}(z,d,u) \tag{.4}$$

This completes the proof.

Proof of Proposition 3. Consider (14). It is solvable iff $(I-P_R(A_4^0))A_3^0p(t)\equiv 0$, and its general solution has the following form (see [Albert, 1972]): $q(t) = -(A_4^0)^+A_3^0p(t) + P_N(A_4^0)v(t), v\in\mathbb{L}_2(t_0,t_1)$. Using this we transform (13)-(14) into the equivalent form ($k=0$):

$$\frac{dp}{dt} = (A_1^k - A_2^k(A_4^k)^+A_3^k)p(t) + A_2^kP_N(A_4^k)v(t), \tag{.5}$$

$$(I-P_R(A_4^k))A_3^kp(t)\equiv 0,\; p(t_1)=w\,. \tag{.6}$$

Define $r_1 := \dim\mathscr{N}((I-P_R(A_4^0))A_3^0)$ and $r_0 := n$. There are three different possibilities: $r_1=r_0$, $r_1=0$ and $0<r_1<r_0$. Consider the case $0<r_1<r_0$. Define

$$\frac{dp^k}{dt} = A_1^k p^k(t) + A_2^k q^k(t), \quad p^k(t_1) = w, \tag{.7}$$
$$0 = A_3^k p^k(t) + A_4^k q^k(t)\,. \tag{.8}$$

We claim that for $k = 1$ any solution of (.7)-(.8) corresponds to some solution of (13)-(14) through (20)-(21), provided $P_0^1 w = w$. Let us prove this. Assume $P_0^1 w = w$ and p^1, q^1 solve (.7)-(.8). Let p^0, q^0 be defined by (20)-(21) with $p^* = p^1$, $q^* = q^1$, $k = 0$. Then, by direct substitution one checks that p^0, q^1 solve (.5)-(.6) so p^0, q^0 solve (13)-(14). On the other hand, if p, v solve (.5)-(.6) then $P_0^1 w = w$ and p, v verify (.7) (this may be checked by direct substitution noting that $P_N((I - P_R(A_4^0))A_3^0)p = p$ by (.6)) and (.8) (one checks this differentiating (.6)).
Now, as above we transform (.7)-(.8) into (.5)-(.6) but with $k = 1$ and define $r_2 := \dim \mathscr{N}((I - P_R(A_4^1))A_3^1)$. Assuming $0 < r_2 < r_1$ we continue constructing (.7)-(.8) and transforming it into (.5)-(.6). At some point for $r_i := \dim \mathscr{N}(I - P_R(A_4^{i-1}))A_3^{i-1}$ should be either $r_i = r_{i-1}$ or $r_i = 0$. Assume $r_i = 0$. Then $\mathscr{N}((I - P_R(A_4^{i-1}))A_3^{i-1}) = \{0\}$ and, thus, $P_N\big[(I - P_R(A_4^{i-1}))A_3^{i-1}\big] = 0_n$ so by (15),(17): $A_1^k \equiv 0_{n\times n}$ and $A_3^k \equiv 0_{m\times n}$ for $k \geq i$. We also have $A_2^i = A_2^{i-1}P_N(A_4^{i-1})$, $A_4^i = A_2^i$ so that $A_2^{i+1} = A_2^i P_N(A_4^i) = 0_n$ and thus $A_4^{i+1} = 0_{m\times n}$ by (18). We see that the statement 1) of the proposition holds for $s = i + 1$. If $r_i = r_{i-1}$ then $\mathscr{N}((I - P_R(A_4^{i-1}))A_3^{i-1}) = \mathbb{R}^n$ so that $P_N\big[(I - P_R(A_4^{i-1}))A_3^{i-1}\big] = I_n$ and thus $A_3^i = 0_{m\times n}$, $A_4^i = 0_{m\times n}$ by (17)-(18) so 1) holds with $s = i$.
Assume 1) is verified for $k = s$. By construction, for any solution p, q of (13)-(14) we have that $p = p^0$ and $q = q^0$ where p^0, q^0 are defined by (20) and (21) with $p^* = p^s$ and $q^* = q^s$, provided p^s, q^s solve (.7)-(.8) and $P_0^s w = w$. Since $A_3^s = A_4^s = 0_{m\times n}$ it follows that (22) is equivalent to (.7)-(.8) for $k = s$ and so statement 3) is verified. By 3) $w = p(t_1) = p^0(t_1) = P_0^s\overline{p}(t_1) = P_0^s w$ so equality $w = P_0^s w$ follows from the solvability of (13)-(14). On the other hand, if $P_0^s w = w$ and $\overline{p}, \overline{q}$ solve (22) then p^0, q^0 defined by (20) and (21) with $p^* = \overline{p}$ and $q^* = \overline{q}$ solve (13)-(14) by construction. This completes the proof.

Proof of Theorem 4. Define $p := P_R(F)z$, $\tilde{q} := P_N(F')z$, $q := (\tilde{q}, u)'$ and let $A_{1,2,3,4}^0$ be defined as above. We note that $F'z = F'p$ and so (11) is equivalent to DAE (13)-(14). By statement 2) of Proposition 3 DAE (13)-(14) is solvable iff $P_0^s(F^+)'F\ell = (F^+)'F\ell$. This and Proposition 2 prove the second line in (23).
As we saw above (11) is equivalent to DAE (13)-(14). Let us represent $\mathscr{I}$ in (12) as a function of p and q. Recalling that $p(t_0) = F^{+'}F'z(t_0)$ we write:

$$\mathscr{I}(z,d,u) = \|Q_0^{-\frac{1}{2}}(p(t_0) - d)\|^2 + \mathscr{I}_1(p + \tilde{q}, \begin{pmatrix} 0 & I \end{pmatrix} q)\,.$$

Define $\tilde{\mathscr{I}_1}(z,u) := \min_{d:F'd=0} \mathscr{I}(z,d,u)$. It is easy to compute that $\min_{d:F'd=0} \|Q_0^{-\frac{1}{2}}(p(t_0) - d)\|^2 = p^T(t_0)\tilde{Q}_0 p(t_0)$. and so

$$\tilde{\mathscr{I}_1}(z,u) = p^T(t_0)\tilde{Q}_0 p(t_0) + \mathscr{I}_1(p + \tilde{q}, \begin{pmatrix} 0 & I \end{pmatrix} q)\,.$$

By statement 3) of Proposition 3 any solution p, q of (13)-(14) may be represented as $p = P_0^s\overline{p}$ and $q = \alpha^s\overline{p} + \beta^s\overline{q}$ where $\overline{p}, \overline{q}$ solve (22). Using this it is straightforward to check that $\tilde{\mathscr{I}_1}(z,u) = \mathscr{I}_2(\overline{p}, \overline{q})$ where:

$$\begin{aligned}\mathscr{I}_2(\overline{p}, \overline{q}) &= \overline{p}^T(t_0)\overline{Q}_0\overline{p}(t_0) + \int_{t_0}^{t_1} \overline{p}^T\overline{Q}\,\overline{p}dt \\ &+ \int_{t_0}^{t_1} (\overline{q}^T\gamma\overline{q} + 2\overline{q}^T\beta^{s\prime}\tilde{Q}B^s\overline{p})dt\,.\end{aligned} \tag{.9}$$

Since $\overline{q}$ uniquely defines $\overline{p}$ through (22), it follows that

$$\min_{z,u \text{ solve } (11), F'd=0} \mathscr{I}(z,d,u) = \min_{\overline{q}} \mathscr{I}_2(\overline{p}, \overline{q}) \tag{.10}$$

Note that $\mathscr{I}_2$ is convex and smooth in $\overline{q}$. Thus, by [Ioffe and Tikhomirov, 1974, p.81,Prop.1] $\hat{q}$ is a minimizer for $\mathscr{I}_2$ iff the Gateaux derivative of $\mathscr{I}_2$ at $\hat{q}$ vanishes: $\frac{d}{d\tau}\mathscr{I}_2(\overline{p}_\tau, \hat{q} + \tau v)\Big|_{\tau=0} = 0$, where $\overline{p}_\tau$ denotes the solution of (22) with $\overline{q} = \hat{q} + \tau v$. Define

$$\begin{aligned}\frac{d\hat{p}}{dt} &= A_1^s\hat{p} + A_2^s\hat{q}, \hat{p}(t_1) = F'^{+}F'\ell, \\ \frac{d\hat{w}}{dt} &= -A_1^{s\prime}\hat{w} + \overline{Q}\hat{p} + B^{s\prime}\tilde{Q}\beta^s\hat{q}, \hat{w}(t_0) = \overline{Q}_0\hat{p}(t_0)\,.\end{aligned} \tag{.11}$$

It is not difficult to check that

$$\begin{aligned}\frac{d}{d\tau}\mathscr{I}_2(\overline{p}_\tau, \hat{q} + \tau v)\Big|_{\tau=0} &= 2\int_{t_0}^{t_1} v^T\gamma\hat{q}dt \\ &+ 2\int_{t_0}^{t_1} v^T\big(\beta^{s\prime}\tilde{Q}B^s\hat{p} - A_2^{s\prime}\hat{w}\big)dt\,.\end{aligned}$$

So the Gateaux derivative of $\mathscr{I}_2$ at $\hat{q}$ vanishes iff $\hat{q}$ solves the linear algebraic equation:

$$\gamma\hat{q} = -\beta^{s\prime}\tilde{Q}B^s\hat{p} + A_2^{s\prime}\hat{w}\,. \tag{.12}$$

We note that, $\det\gamma = 0$ in the general case. So (.12) is solvable iff the vector on the right hand side of (.12) belongs to the range $\mathscr{R}(\gamma)$ of matrix γ. As $\tilde{Q}$ is a symmetric positive definite matrix, then $\mathscr{R}(\gamma) = \mathscr{R}(\beta^{s\prime})$. It is clear that $-\beta^{s\prime}\tilde{Q}B^s\hat{p} \in \mathscr{R}(\beta^{s\prime})$. By definition, $\beta^{s\prime} = P_N(A_4^{s-1}) \times \cdots \times P_N(A_4^0)$ and $A_2^{s\prime} = P_N(A_4^{s-1}) \times \cdots \times P_N(A_4^0)A_2^0$ so $A_2^{s\prime}\hat{w} \in \mathscr{R}(\beta^{s\prime})$. Thus, the minimizer for $\mathscr{I}_2$ admits representation:

$$\hat{q} = \gamma^+\big(A_2^{s\prime}\hat{w} - \beta^{s\prime}\tilde{Q}B^s\hat{p}\big) + v, \tag{.13}$$

where $\gamma v = 0$. Let us plug (.13) into (.11). Then $\gamma v = 0$ implies $A_2^s v = 0$. Thus $\hat{p}$, $\hat{w}$ do not depend on v and so does $\mathscr{I}_2$. Therefore, we can take $v = 0$ in (.13). Now we split (.11) using the standard argument of the linear regulator problem (see [Fleming and Rishel, 1975, p.23,Exmpl.2.3,p.88]): plugging (.13) with $v = 0$ into (.11) we find that $\hat{w} = K\hat{p}$ and $\hat{q} = \gamma^+\big(A_2^{s\prime}K - \beta^{s\prime}\tilde{Q}B^s\big)\hat{p}$ where K solves Riccati equation formulated in the theorem's statement. Thus, $\hat{q}$ is in the feed-back form. In order to compute ℓ-estimate $\hat{u}$ which solves (11), we recall that $u = \begin{pmatrix} 0 & I \end{pmatrix} q$ and $q = \alpha^s\overline{p} + \beta^s\overline{q}$ so that $\hat{u} = \begin{pmatrix} 0 & I \end{pmatrix}(\alpha^s\hat{p} + \beta^s\hat{q})$. Using this representation and definition of $\hat{x}$ we find integrating by parts that: $\hat{u}(y) = \int_{t_0}^{t_1} \hat{q}^T\binom{0}{y}dt = \ell^T FF^+\hat{x}(t_1)$. Let us compute ℓ-error. We find plugging (.11) into (.9) and using (.4), (.10) that $\hat{\sigma}(t_1,\ell) = \mathscr{I}_2(\hat{p}, \hat{q}) = \hat{p}^T(t_1)FF^+\ell = \ell^T FF^+K(t_1)FF^+\ell$. This completes the proof.