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Well-posedness of a conservation law with non-local flux arising in traffic flow modeling

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Abstract We prove the well-posedness of entropy weak solutions of a scalar conservation law with non-local flux arising in traffic flow modeling. The result is obtained providing accurate $\mathbf{L}^{\infty}$, BV and $\mathbf{L}^{\mathbf{1}}$ estimates for the sequence of approximate solutions constructed by an adapted Lax-Friedrichs scheme.

Keywords Scalar conservation laws · Non-local flux · Lax-Friedrichs scheme · Macroscopic Traffic Models

1 Introduction

Conservation laws with non-local terms arise in a variety of physical applications. Space-integral terms are considered for example in models for granular flows [1], sedimentation [6], crowd motion [9], or more general problems like gradient constrained equations [2]. Also, non-local in time terms arise in conservation laws with memory, starting from [13].

Macroscopic traffic flow models usually consist of one or two first-order hyperbolic partial differential equations accounting for the conservation of the number of cars, as in the celebrated Lighthill-Whitham-Richards model [20, 21], and for a momentum conservation or balance equation, see for example the Aw-Rascle model [4], the general phase transition model [7], or the generalized

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second order model [18]. The resulting (system of) classical conservation laws usually turns out to be well-posed.

Equations with non-local flux have been recently introduced in traffic flow modeling to account for the reaction of drivers or pedestrians to the surrounding density of other individuals, see [3, 9–12, 22]. While pedestrians are likely to react to the presence of people all around them, drivers will mainly adapt their velocity to the downstream traffic, assigning a greater importance to closer vehicles. In this paper, we consider the following mass conservation equation for traffic flow with non-local mean velocity:

$$\partial_t \rho(t,x) + \partial_x \left(\rho(t,x) v\big(\int_x^{x+\eta} \rho(t,y) w_\eta(y-x)\ dy \big) \right) = 0\ , \tag{1}$$

defined for $t \in \mathbb{R}^+$ and $x \in \mathbb{R}$, $\eta > 0$. Above, the convolution kernel $w_\eta \in \mathbf{C^1}([0,\eta];\mathbb{R}^+)$ is a non-increasing function such that $\int_0^\eta w_\eta(x) dx = 1$ (for example, $w_\eta(x) \equiv 1/\eta$ or $w_\eta(x) = 2(1 - x/\eta)/\eta$).
With slight abuse of notation, we define the downstream convolution product as

$$\rho *_d w_\eta(t,x) := \int_x^{x+\eta} \rho(t,y) w_\eta(y-x)\ dy\ . \tag{2}$$

This choice is intended to model the behavior of drivers reacting to what happens in front of them, thus adapting their velocity with respect to the downstream density.

For the sake of simplicity, and aiming to get sharp estimates, in this paper we take the mean velocity function to be $v(r) = 1 - r$. The same approach could be applied to more general continuous decreasing functions.

Setting $V(t,x) = v(\rho *_d w_\eta(t,x))$, we rewrite (1) as

$$\partial_t \rho(t,x) + \partial_x \left(\rho(t,x) V(t,x) \right) = 0\ ,$$

and we couple it with an initial datum

$$\rho(0,x) = \rho_0(x) \in \mathrm{BV}(\mathbb{R};[0,1]). \tag{3}$$

We will consider solutions $\rho = \rho(t,x)$ satisfying the following definition.

Definition 1 A function $\rho \in (\mathbf{L^1} \cap \mathbf{L^\infty} \cap \mathrm{BV})(\mathbb{R}^+ \times \mathbb{R};\mathbb{R})$ is a weak solution of (1), (3), if

$$\int_0^{+\infty}\int_{-\infty}^{+\infty} \left(\rho\varphi_t + \rho(t,x) v(\rho *_d w_\eta) \varphi_x \right) dxdt + \int_{-\infty}^{+\infty} \rho_0(x)\varphi(0,x)\ dx = 0 \tag{4}$$

for all $\varphi \in \mathbf{C^1_c}(\mathbb{R}^2;\mathbb{R})$.

Following [17, Definition 1], [6, Definition 4.1] and [10, Definition 2.1], we can also give an entropy criterion to select admissible solutions.

Definition 2 A function $\rho \in (\mathbf{L^1} \cap \mathbf{L^\infty} \cap \mathrm{BV})(\mathbb{R}^+ \times \mathbb{R}; \mathbb{R})$ is an entropy weak solution if

$$\int_0^{+\infty} \int_{-\infty}^{+\infty} \left(|\rho - \kappa|\varphi_t + |\rho - \kappa|\, V\, \varphi_x - \mathrm{sgn}(\rho - \kappa)\, \kappa\, V_x\, \varphi\right)(t, x) dx dt \\ + \int_{-\infty}^{+\infty} |\rho_0(x) - \kappa| \varphi(0, x)\, dx \geq 0 \quad (5)$$

for all $\varphi \in \mathbf{C^1_c}(\mathbb{R}^2; \mathbb{R}^+)$ and $\kappa \in \mathbb{R}$.

The main results of this paper are collected in the following theorem.

Theorem 1 *Let $\rho_0 \in BV(\mathbb{R}; [0, 1])$ and $w_\eta \in \mathbf{C^1}([0, \eta]; \mathbb{R}^+)$ be a non-increasing function such that $\int_0^\eta w_\eta(x) dx = 1$. Then the Cauchy problem*

$$\begin{cases} \partial_t \rho + \partial_x \left(\rho v(\rho *_d w_\eta)\right) = 0, & x \in \mathbb{R}, t > 0, \\ \rho(0, x) = \rho_0(x), & x \in \mathbb{R}, \end{cases}$$

admits a unique weak entropy solution in the sense of Definitions 1 and 2, such that

$$\min_{\mathbb{R}} \{\rho_0\} \leq \rho(t, x) \leq \max_{\mathbb{R}} \{\rho_0\}, \quad \textit{for a.e. } x \in \mathbb{R},\ t > 0. \quad (6)$$

The existence of a weak entropy solution is proved by constructing a converging sequence of finite volume approximate solutions, defined using an adapted Lax-Friedrichs scheme. Compared to the previous studies [3, 6, 9–11], our result does not require the kernel w_η to be smooth, and provides very accurate $\mathbf{L^\infty}$ bounds on the solutions (which are given by the maximum principle (6) above). Moreover, equation (1) provides a meaningful model of road traffic evolution.

The paper is organized as follows. Section 2 contains the description of the finite volume scheme used to construct approximate solutions, and the proofs of its fine properties: maximum principle, bounded total variation, discrete entropy inequalities and $\mathbf{L^1}$ stability estimates. Relying on these results, the proof of Theorem 1 is detailed in Section 3. Some numerical simulations illustrating the properties of the solutions are collected in Section 4.

2 Uniqueness of entropy solutions

Uniqueness of entropy solutions in the sense of Definition 2 follows from their Lipschitz continuous dependence with respect to the flux function, as in [6, 16]. Indeed, we get the following result:

Theorem 2 *Let ρ, σ be two entropy solutions to* (1), (3) *with initial data ρ_0, σ_0 respectively. Then, for any $T > 0$ there holds*

$$\|\rho(t, \cdot) - \sigma(t, \cdot)\|_{\mathbf{L^1}} \leq e^{\mathcal{K}T} \|\rho_0 - \sigma_0\|_{\mathbf{L^1}} \qquad \forall t \in (0, T], \quad (7)$$

where the constant $\mathcal{K}$ is given by (11).

Proof We follow [16, Theorem 1.3] and [6, Theorem 4.1]. The functions ρ and σ are respectively entropy solutions of

$$\begin{aligned}\partial_t\rho(t,x)+\partial_x\left(\rho(t,x)V(t,x)\right)=0\,,\quad & V=v(\rho*w_\eta)\,,\quad & \rho(0,x)=\rho_0(x)\,,\\ \partial_t\sigma(t,x)+\partial_x\left(\sigma(t,x)U(t,x)\right)=0\,,\quad & U=v(\sigma*w_\eta)\,,\quad & \sigma(0,x)=\sigma_0(x)\,.\end{aligned}$$

We remark here that, since $\rho,\sigma\in(\mathbf{L^1}\cap\mathbf{L}^\infty\cap\mathrm{BV})(\mathbb{R}^+\times\mathbb{R};\mathbb{R})$, both V and U are bounded measurable functions and are Lipschitz continuous w.r. to x. Indeed, we have

$$\|V_x\|_\infty\le 2w_\eta(0)\|v'\|_\infty\|\rho\|_\infty,\quad \|U_x\|_\infty\le 2w_\eta(0)\|v'\|_\infty\|\sigma\|_\infty.$$

We then proceed using the classical doubling of variables technique introduced by Kružkov [17], see also [6,16], which gives the following inequality

$$\begin{aligned}\|\rho(T,\cdot)-\sigma(T,\cdot)\|_{\mathbf{L^1}}\le\ &\|\rho_0-\sigma_0\|_{\mathbf{L^1}}\\ &+\int_0^T\int_{\mathbb{R}}|\rho_x(t,x)||U(t,x)-V(t,x)|\;dxdt\\ &+\int_0^T\int_{\mathbb{R}}|\rho(t,x)||U_x(t,x)-V_x(t,x)|\;dxdt,\end{aligned}\tag{8}$$

for $T>0$. We observe that

$$|U(t,x)-V(t,x)|\le w_\eta(0)\|v'\|_\infty\|\rho(t,\cdot)-\sigma(t,\cdot)\|_{\mathbf{L^1}}\tag{9}$$

and that for a. e. $x\in\mathbb{R}$

$$\begin{aligned}&|U_x(t,x)-V_x(t,x)|\\ &\le|v'(\rho*w_\eta)-v'(\sigma*w_\eta)|\\ &\times\left|-\int_x^{x+\eta}\rho(t,y)w_\eta'(y-x)\,dy+\rho(t,x+\eta)w_\eta(\eta)-\rho(t,x)w_\eta(0)\right|\\ &\quad+|v'(\sigma*w_\eta)|\\ &\times\left|\int_x^{x+\eta}(\sigma(t,y)-\rho(t,y))w_\eta'(y-x)\,dy+(\rho-\sigma)(t,x+\eta)w_\eta(\eta)-(\rho-\sigma)(t,x)w_\eta(0)\right|\\ &\le\left(\;2(w_\eta(0))^2\|v''\|_\infty\|\rho(t,\cdot)\|_\infty+\|v'\|_\infty\left\|w_\eta'\right\|_{\mathbf{L}^\infty([0,\eta])}\right)\|\rho(t,\cdot)-\sigma(t,\cdot)\|_1\\ &\quad+w_\eta(0)\|v'\|_\infty\left(|\rho-\sigma|(t,x+\eta)+|\rho-\sigma|(t,x)\right).\end{aligned}\tag{10}$$

Plugging (9), (10) into (8) we get

$$\|\rho(T,\cdot)-\sigma(T,\cdot)\|_{\mathbf{L^1}}\le\|\rho_0-\sigma_0\|_{\mathbf{L^1}}+\mathcal{K}\int_0^T\|\rho(t,\cdot)-\sigma(t,\cdot)\|_{\mathbf{L^1}}dt$$

with

$$\mathcal{K}=\ w_\eta(0)\|v'\|_\infty\left(\sup_{t\in[0,T]}\|\rho(t,\cdot)\|_{\mathrm{BV}(\mathbb{R})}+2\sup_{t\in[0,T]}\|\rho(t,\cdot)\|_\infty\right)$$

$$+ \sup_{t\in[0,T]} \|\rho(t,\cdot)\|_1 \left(2(w_\eta(0))^2 \|v''\|_\infty \sup_{t\in[0,T]} \|\rho(t,\cdot)\|_\infty + \|v'\|_\infty \left\|w_\eta'\right\|_{\mathbf{L}^\infty([0,\eta])} \right). \tag{11}$$

By Gronwall's lemma, we get (7).

Remark 1 By Proposition 1 in Section 3.1, we will prove that solutions to (1), (3) satisfy a maximum principle. In particular, $\sup_{t\in[0,T]} \|\rho(t,\cdot)\|_\infty = \|\rho_0\|_\infty$ and $\rho(t,\cdot) \geq 0$ for all $t \in [0,T]$. Therefore, by mass conservation we have also $\sup_{t\in[0,T]} \|\rho(t,\cdot)\|_1 = \|\rho_0\|_1$ and the constant in (11) can be rewritten as

$$\begin{aligned} \mathcal{K} = \; & w_\eta(0)\|v'\|_\infty \left(\sup_{t\in[0,T]} \|\rho(t,\cdot)\|_{\mathrm{BV}(\mathbb{R})} + 2\|\rho_0\|_\infty \right) \\ & + \|\rho_0\|_1 \left(2(w_\eta(0))^2 \|v''\|_\infty \|\rho_0\|_\infty + \|v'\|_\infty \left\|w_\eta'\right\|_{\mathbf{L}^\infty([0,\eta])} \right). \end{aligned}$$

3 A Lax-Friedrichs numerical scheme

We take a space step Δx such that $\eta = N\Delta x$, for some $N \in \mathbb{N}$, and a time step Δt subject to a CFL condition to be specified later. For $j \in \mathbb{Z}$ and $n \in \mathbb{N}$, let $x_{j+1/2} = j\Delta x$ be the cells interfaces, $x_j = (j-1/2)\Delta x$ the cells centers and $t^n = n\Delta t$ the time mesh. We want to construct a finite volume approximate solution $\rho_{\Delta x}(t,x) = \rho_j^n$ for $(t,x) \in C_j^n = [t^n, t^{n+1}[\times]x_{j-1/2}, x_{j+1/2}[$. To this end, we approximate the initial datum ρ_0 with the piecewise constant function given by

$$\rho_j^0 = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} \rho_0(x)\; dx.$$

Moreover, we denote $w_\eta^k := w_\eta(k\Delta x)$ for $k = 0, \ldots, N-1$ and set

$$V_j^n := 1 - \Delta x \sum_{k=0}^{N-1} w_\eta^k \; \rho_{j+k}^n, \tag{12}$$

which involves a quadrature formula to approximate the convolution term. Remark that the above discretization choice for w_η implies

$$\Delta x \sum_{k=0}^{N-1} w_\eta^k \leq 1 + w_\eta(0)\Delta x. \tag{13}$$

We consider the following modified Lax-Friedrichs flux adapted to (1):

$$\begin{aligned} F_{j+1/2}^n :=& g(\rho_j^n, \ldots, \rho_{j+N}^n) \\ =& \frac{1}{2}\rho_j^n V_j^n + \frac{1}{2}\rho_{j+1}^n V_{j+1}^n + \frac{\alpha}{2}(\rho_j^n - \rho_{j+1}^n)\;, \end{aligned} \tag{14}$$

where $\alpha \geq 1$ is the viscosity coefficient. This gives the following $N+2$ points finite volume scheme

$$\rho_j^{n+1} = H(\rho_{j-1}^n, \dots, \rho_{j+N}^n) \ , \tag{15}$$

where

$$\begin{aligned} & H(\rho_{j-1}, \dots, \rho_{j+N}) \\ & := \rho_j - \lambda \left(g(\rho_j, \dots, \rho_{j+N}) - g(\rho_{j-1}, \dots, \rho_{j+N-1}) \right) \\ & = \rho_j + \frac{\lambda \alpha}{2} \left(\rho_{j-1} - 2\rho_j + \rho_{j+1} \right) + \frac{\lambda}{2} \left(\rho_{j-1} V_{j-1} - \rho_{j+1} V_{j+1} \right), \end{aligned} \tag{16}$$

with $\lambda = \Delta t / \Delta x$. In particular, we observe that $H(\rho, \dots, \rho) = \rho$ for all $\rho \in [0,1]$.

Assuming $\rho_i \in [0,1]$ for $i = j-1, \dots, j+N$, straightforward computations give:

$$\frac{\partial H}{\partial \rho_{j-1}} = \frac{\lambda}{2} \Big(\alpha + 1 - \Delta x \sum_{k=0}^{N-1} w_\eta^k \ \rho_{j+k-1} - \Delta x \, w_\eta^0 \, \rho_{j-1} \Big) \tag{17a}$$

$$\frac{\partial H}{\partial \rho_j} = 1 - \lambda \left(\alpha + \frac{1}{2} \Delta x \, w_\eta^1 \ \rho_{j-1} \right) \geq 1 - \lambda \left(\alpha + \Delta x \, \frac{w_\eta(0)}{2} \right) \tag{17b}$$

$$\frac{\partial H}{\partial \rho_{j+1}} = \frac{\lambda}{2} \Big(\alpha - 1 + \Delta x \sum_{k=0}^{N-1} w_\eta^k \ \rho_{j+k+1} - \Delta x \, w_\eta^2 \, \rho_{j-1} + \Delta x \, w_\eta^0 \, \rho_{j+1} \Big) \tag{17c}$$

$$\frac{\partial H}{\partial \rho_{j+k}} = \frac{\lambda}{2} \ \Delta x \left(w_\eta^{k-1} \rho_{j+1} - w_\eta^{k+1} \rho_{j-1} \right) \qquad k = 2, \dots, N-2 \tag{17d}$$

$$\frac{\partial H}{\partial \rho_{j+N-1}} = \frac{\lambda}{2} \ \Delta x \ w_\eta^{N-2} \rho_{j+1} \tag{17e}$$

$$\frac{\partial H}{\partial \rho_{j+N}} = \frac{\lambda}{2} \ \Delta x \ w_\eta^{N-1} \rho_{j+1} \tag{17f}$$

Observe that (17e) and (17f) are non-negative. Moreover, the CFL condition

$$\Delta t \leq \frac{2}{2\alpha + \Delta x \, w_\eta(0)} \ \Delta x \tag{18}$$

ensures the positivity of (17b) and the assumption

$$\alpha \geq 1 + \Delta x \ w_\eta(0) \tag{19}$$

guarantees the increasing monotonicity w.r.t. ρ_{j+1} in (17c), and combined with (13), guarantees the non-negativity of (17a). To obtain (18) and (19) we used the fact that $w_\eta^k \leq w_\eta(0)$ for all $k = 0, \dots, N-1$, by non-increasing monotonicity assumption. On the contrary, the sign of (17d) cannot be a-priori determined. (Nevertheless, we observe that if $\rho_{j-1} = \rho_{j+1}$ the derivatives in (17d) are non-negative due to the decreasing monotonicity of w_η.) Therefore, the numerical scheme (14), (15) is not monotone, and classical convergence results do not apply. Nevertheless, we are able to recover the necessary $\mathbf{L}^\infty$ and BV bounds.

3.1 Maximum principle and $\mathbf{L}^\infty$ estimates

The sought $\mathbf{L}^\infty$ bound is a direct consequence of a maximum principle property.

Proposition 1 *For any initial datum ρ_j^0, $j \in \mathbb{Z}$, let $\rho_m = \min_{j\in\mathbb{Z}}\{\rho_j^0\} \in [0,1]$ and $\rho_M = \max_{j\in\mathbb{Z}}\{\rho_j^0\} \in [0,1]$. Then the finite volume approximation ρ_j^n, $j \in \mathbb{Z}$ and $n \in \mathbb{N}$, constructed using scheme* (14), (15) *satisfies the bounds*

$$\rho_m \leq \rho_j^n \leq \rho_M$$

for all $j \in \mathbb{Z}$ and $n \in \mathbb{N}$, under the CFL condition (18).

The proof is based on the following lemma.

Lemma 1 *Let $0 \leq \rho_m \leq \rho_j^n \leq \rho_M \leq 1$ for all $j \in \mathbb{Z}$. Then*

$$H(\rho_m, \rho_m, \rho_m, \rho_{j+2}, \ldots, \rho_{j+N-2}, \rho_m, \rho_m) \geq \rho_m, \tag{20}$$

$$H(\rho_M, \rho_M, \rho_M, \rho_{j+2}, \ldots, \rho_{j+N-2}, \rho_M, \rho_M) \leq \rho_M. \tag{21}$$

Proof From (16) we get

$$H(\rho_m, \rho_m, \rho_m, \rho_{j+2}, \ldots, \rho_{j+N-2}, \rho_m, \rho_m) = \rho_m + \frac{\Delta t}{2}\rho_m \sum_{k=0}^{N-1} w_\eta^k \left(\rho_{j+k+1} - \rho_{j+k-1}\right),$$

and we observe that

$$\begin{aligned}
&\sum_{k=0}^{N-1} w_\eta^k \left(\rho_{j+k+1} - \rho_{j+k-1}\right) \\
&\quad = \rho_m \left[w_\eta^{N-2} + w_\eta^{N-1} - w_\eta^0 - w_\eta^1\right] + \sum_{k=1}^{N-2} \rho_{j+k} \left[w_\eta^{k-1} - w_\eta^{k+1}\right] \\
&\quad \geq \rho_m \left[w_\eta^{N-2} + w_\eta^{N-1} - w_\eta^0 - w_\eta^1\right] + \rho_m \sum_{k=1}^{N-2} \left[w_\eta^{k-1} - w_\eta^{k+1}\right] \\
&\quad = \rho_m \Big\{ \sum_{k=1}^{N} w_\eta^{k-1} - \sum_{k=-1}^{N-2} w_\eta^{k+1} \Big\} = 0 \ ,
\end{aligned}$$

where the inequality is due to the non-increasing monotonicity of w_η. Inequality (21) can be recovered following the same procedure.

Proof of Proposition 1 We apply the mean value theorem between the points $R_j^n = (\rho_{j-1}^n, \ldots, \rho_{j+N}^n)$ and

$$R_m^n = (\rho_m, \rho_m, \rho_m, \rho_{j+2}^n, \ldots, \rho_{j+N-2}^n, \rho_m, \rho_m),$$

which by (20) tells us

$$\rho_j^{n+1} = H(R_j^n) = H(R_m^n) + \left\langle \nabla H(R_\xi), R_j^n - R_m^n \right\rangle$$

$$\geq \rho_m + \left\langle \nabla H(R_\xi), R_j^n - R_m^n \right\rangle , \tag{22}$$

for $R_\xi = (1-\xi)R_m^n + \xi R_j^n$, for some $\xi \in [0,1]$.
It is now enough to observe that

$$\frac{\partial H}{\partial \rho_{j+k}}(R_\xi)(R_j^n - R_m^n)_k = 0 \qquad k = 2, \ldots, N-2,$$

since $(R_j^n - R_m^n)_k = 0$ for $k = 2, \ldots, N-2$. Therefore, under the assumptions (18) and (19), we can conclude that $\left\langle \nabla H(R_\xi), R_j^n - R_m^n \right\rangle \geq 0$ and therefore by (22) we have proved that

$$\rho_j^{n+1} \geq \rho_m.$$

The upper bound $\rho_j^{n+1} \leq \rho_M$ is recovered smilarly by considering

$$R_M^n = (\rho_M, \rho_M, \rho_M, \rho_{j+2}^n, \ldots, \rho_{j+N-2}^n, \rho_M, \rho_M)$$

in place of R_m^n and using (21).

Remark 2 We note that, by (12) and the above maximum principle, for any initial datum ρ_0 such that $\rho_j^0 \in [0,1]$ for all $j \in \mathbb{Z}$, the approximated velocity V_j^n satisfy the bounds $-w_\eta(0)\Delta x \leq V_j^n \leq 1$ for all $j \in \mathbb{Z}$, $n \in \mathbb{N}$. Therefore, our choice of the discretization of the convolution integral is easy to implement but slightly underestimates traffic mean velocity and may introduce unphysical negative velocities. Other discretization choices are possible, see for example [6, eq. (3.2)].

3.2 BV estimates

BV estimates cannot be derived here using standard general approaches. Indeed, scheme (15), (16) does not fit the classical assumption of TVD schemes, see [15, Lemma 2.2], and [5] for the generalization of these conditions to N-points schemes. Nevertheless, accurate estimates show that the approximate solutions constructed using our numerical scheme have bounded total variation and preserve monotonicity.

Proposition 2 *Let $\rho_0 \in BV(\mathbb{R};[0,1])$, and let $\rho_{\Delta x}$ be given by* (15), (16). *If $\alpha \geq 1 + 2\,\Delta x\, w_\eta(0)$ and the CFL condition $\Delta t \leq 2\Delta x/(2\alpha + 3\Delta x\, w_\eta(0))$ holds, then for every $T > 0$ the following discrete space BV estimate is satisfied*

$$TV(\rho_{\Delta x}(T,\cdot)) \leq C(w_\eta, \rho_0, T) := e^{2w_\eta(0)T}\, TV(\rho_0). \tag{23}$$

In particular, the numerical scheme (15), (16) *is monotonicity preserving.*

Proof Scheme (15), (16), can be rewritten as

$$\rho_j^{n+1} = \rho_j^n - \frac{\lambda}{2}\Big(\alpha + 1 - \Delta x \sum_{k=0}^{N-1} w_\eta^k\, \rho_{j+k+1}^n - \Delta x\, w_\eta^0\, \rho_{j-1}^n\Big)\Delta_{j-1/2}^n$$

$$+\frac{\lambda}{2}\Big(\alpha-1+\Delta x\sum_{k=0}^{N-1}w_\eta^k\ \rho^n_{j+k+1}+\Delta x\,(w^0_\eta+w^1_\eta)\,\rho^n_{j-1}\Big)\Delta^n_{j+1/2}$$
$$+\frac{\lambda}{2}\rho^n_{j-1}\Delta x\sum_{k=2}^{N-1}(w_\eta^{k-1}+w_\eta^k)\ \Delta^n_{j+k-1/2}$$
$$+\frac{\lambda}{2}\Delta x\ w_\eta^{N-1}\ \rho^n_{j-1}\ \Delta^n_{j+N-1/2}\ ,$$

where we have set $\Delta^n_{j+k-1/2}=\rho^n_{j+k}-\rho^n_{j+k-1}$ for $k=0,\dots,N$.
In the same way we get

$$\begin{aligned}\rho^{n+1}_{j+1}=\ &\rho^n_{j+1}-\frac{\lambda}{2}\Big(\alpha+1-\Delta x\sum_{k=0}^{N-1}w_\eta^k\ \rho^n_{j+k+2}-\Delta x\,w^0_\eta\,\rho^n_j\Big)\Delta^n_{j+1/2}\\&+\frac{\lambda}{2}\Big(\alpha-1+\Delta x\sum_{k=0}^{N-1}w_\eta^k\ \rho^n_{j+k+2}+\Delta x\,(w^0_\eta+w^1_\eta)\,\rho^n_j\Big)\Delta^n_{j+3/2}\\&+\frac{\lambda}{2}\rho^n_j\,\Delta x\sum_{k=2}^{N-1}(w_\eta^{k-1}+w_\eta^k)\ \Delta^n_{j+k+1/2}\\&+\frac{\lambda}{2}\Delta x\ w_\eta^{N-1}\ \rho^n_j\ \Delta^n_{j+N+1/2}\ .\end{aligned}$$

Therefore computing the difference gives

$$\begin{aligned}&\Delta^{n+1}_{j+1/2}=\\&=\frac{\lambda}{2}\Big(\alpha+1-\Delta x\sum_{k=0}^{N-1}w_\eta^k\ \rho^n_{j+k+1}-\Delta x\,w^0_\eta\,\rho^n_{j-1}\Big)\Delta^n_{j-1/2}\\&+\left[1-\frac{\lambda}{2}\Big(2\alpha-\Delta x\,w^0_\eta\,\rho^n_j+\Delta x\,(w^0_\eta+w^1_\eta)\rho^n_{j-1}-\Delta x\sum_{k=0}^{N-1}w_\eta^k\ \Delta_{j+k+3/2}\Big)\right]\Delta^n_{j+1/2}\\&+\frac{\lambda}{2}\Big(\alpha-1+\Delta x\,(w^0_\eta+w^1_\eta)\,\rho^n_j-\Delta x\,(w^1_\eta+w^2_\eta)\,\rho^n_{j-1}+\Delta x\sum_{k=0}^{N-1}w_\eta^k\ \rho^n_{j+k+2}\Big)\Delta^n_{j+3/2}\\&+\frac{\lambda}{2}\Delta x\sum_{k=2}^{N-2}[(w_\eta^{k-1}+w_\eta^k)\rho^n_j-(w_\eta^k+w_\eta^{k+1})\rho^n_{j-1}]\ \Delta^n_{j+k+1/2}\\&+\frac{\lambda}{2}\Delta x\ [(w_\eta^{N-2}+w_\eta^{N-1})\rho^n_j-w_\eta^{N-1}\rho^n_{j-1}]\Delta^n_{j+N-1/2}\\&+\frac{\lambda}{2}\Delta x\,w_\eta^{N-1}\ \rho^n_j\ \Delta^n_{j+N+1/2}\ .\end{aligned}$$

Adding and subtracting $(w^k_\eta+w^{k+1}_\eta)\rho^n_j$ in the fourth term of the above right-hand side, and noting that

$$[(w_\eta^{N-2}+w_\eta^{N-1})\rho^n_j-w_\eta^{N-1}\rho^n_{j-1}]\Delta^n_{j+N-1/2}=[w_\eta^{N-2}\rho^n_j+w_\eta^{N-1}\Delta^n_{j-1/2}]\Delta^n_{j+N-1/2},$$

we get

$$\Delta^{n+1}_{j+1/2}=$$

$$
\begin{aligned}
&= \frac{\lambda}{2}\Big(\alpha+1-\Delta x\, w_\eta^0\,(\rho_{j-1}^n+\rho_{j+1}^n)-\Delta x\,(w_\eta^1+w_\eta^2)\,\rho_{j+2}^n-\Delta x\sum_{k=2}^{N-2} w_\eta^{k+1}\ \rho_{j+k}^n\Big)\Delta_{j-1/2}^n\\
&\quad+\left[1-\frac{\lambda}{2}\Big(2\alpha-\Delta x\, w_\eta^0\,\rho_j^n+\Delta x\,(w_\eta^0+w_\eta^1)\,\rho_{j-1}^n-\Delta x\sum_{k=0}^{N-1} w_\eta^k\ \Delta_{j+k+3/2}\Big)\right]\Delta_{j+1/2}^n\\
&\quad+\frac{\lambda}{2}\Big(\alpha-1+\Delta x\,(w_\eta^0+w_\eta^1)\,\rho_j^n-\Delta x\,(w_\eta^1+w_\eta^2)\,\rho_{j-1}^n+\Delta x\sum_{k=0}^{N-1} w_\eta^k\ \rho_{j+k+2}^n\Big)\Delta_{j+3/2}^n\\
&\quad+\frac{\lambda}{2}\rho_j^n\,\Delta x\sum_{k=2}^{N-2}(w_\eta^{k-1}-w_\eta^{k+1})\ \Delta_{j+k+1/2}^n\\
&\quad+\frac{\lambda}{2}\Delta x\ w_\eta^{N-2}\rho_j^n\,\Delta_{j+N-1/2}^n+\frac{\lambda}{2}\Delta x\ w_\eta^{N-1}\ \rho_j^n\ \Delta_{j+N+1/2}^n\,.
\end{aligned}
$$

Observe that the first coefficient in the summation is non-negative for Δx sufficiently small such that $\alpha \geq 2w_\eta(0)\Delta x$, since

$$
\begin{aligned}
&\Delta x\, w_\eta^0\,(\rho_{j-1}^n+\rho_{j+1}^n)+\Delta x(w_\eta^1+w_\eta^2)\rho_{j+2}^n+\Delta x\sum_{k=2}^{N-2} w_\eta^{k+1}\ \rho_{j+k}^n\\
&\leq \Delta x\, w_\eta^0+\Delta x\sum_{k=0}^{N-1} w_\eta^k\\
&\leq 1+2w_\eta(0)\Delta x
\end{aligned}
$$

by (13). The second coefficient in the summation is non-negative under the slightly stronger CFL assumption

$$
\Delta t \leq \frac{2}{2\alpha+3\Delta x\, w_\eta(0)}\ \Delta x,
$$

and the third term is non-negative if $\alpha \geq 1+2\,\Delta x\, w_\eta(0)$. Hence all the above coefficients are non-negative, and the above formula guarantees in particular that the scheme (15), (16) is monotonicity preserving. Taking the absolute values in the above expression and summing over $j\in\mathbb{Z}$ we get

$$
\begin{aligned}
&\sum_j \left|\Delta_{j+1/2}^{n+1}\right| =\\
&\leq \sum_j \frac{\lambda}{2}\Big(\alpha+1-\Delta x\, w_\eta^0\,(\rho_{j-1}^n+\rho_{j+1}^n)-\Delta x\,(w_\eta^1+w_\eta^2)\,\rho_{j+2}^n-\Delta x\sum_{k=2}^{N-2} w_\eta^{k+1}\ \rho_{j+k}^n\Big)\left|\Delta_{j-1/2}^n\right|\\
&\quad+\sum_j\left[1-\frac{\lambda}{2}\Big(2\alpha-\Delta x\, w_\eta^0\,\rho_j^n+\Delta x\,(w_\eta^0+w_\eta^1)\,\rho_{j-1}^n-\Delta x\sum_{k=0}^{N-1} w_\eta^k\ \Delta_{j+k+3/2}\Big)\right]\left|\Delta_{j+1/2}^n\right|\\
&\quad+\sum_j\frac{\lambda}{2}\Big(\alpha-1+\Delta x\,(w_\eta^0+w_\eta^1)\,\rho_j^n-\Delta x\,(w_\eta^1+w_\eta^2)\,\rho_{j-1}^n+\Delta x\sum_{k=0}^{N-1} w_\eta^k\ \rho_{j+k+2}^n\Big)\left|\Delta_{j+3/2}^n\right|\\
&\quad+\sum_j\frac{\lambda}{2}\rho_j^n\,\Delta x\sum_{k=2}^{N-2}(w_\eta^{k-1}-w_\eta^{k+1})\ \left|\Delta_{j+k+1/2}^n\right|\\
&\quad+\sum_j\frac{\lambda}{2}\Delta x\ w_\eta^{N-2}\rho_j^n\left|\Delta_{j+N-1/2}^n\right|
\end{aligned}
$$

$$+\sum_j \frac{\lambda}{2}\Delta x\, w_\eta^{N-1}\; \rho_j^n\; \left|\Delta_{j+N+1/2}^n\right| .$$

Rearranging the indices we obtain

$$\begin{aligned}
&\sum_j \left|\Delta_{j+1/2}^{n+1}\right| \\
&\le \sum_j \left|\Delta_{j+1/2}^n\right| \Bigg[1+\frac{\Delta t}{2}\Bigg(\sum_{k=2}^{N-2}(w_\eta^{k-1}-w_\eta^{k+1})\rho_{j+k+1}^n-(w_\eta^1+w_\eta^2)\rho_{j-2}^n \\
&\quad+\sum_{k=2}^{N-2}(w_\eta^{k-1}-w_\eta^{k+1})\rho_{j-k}^n-(w_\eta^1+w_\eta^2)\rho_{j+3}^n+w_\eta^{N-2}\rho_{j+N}^n \\
&\quad+w_\eta^{N-1}\rho_{j+N+1}^n+w_\eta^{N-2}\rho_{j-N-1}^n+w_\eta^{N-1}\;\rho_{j-N}^n\Bigg)\Bigg] \\
&\le \left[1+\frac{\Delta t}{2}\left(2\sum_{k=2}^{N-2}(w_\eta^{k-1}-w_\eta^{k+1})+2w_\eta^{N-2}+2w_\eta^{N-1}\right)\right]\sum_j\left|\Delta_{j+1/2}^n\right| \\
&\le (1+2w_\eta(0)\Delta t)\sum_j\left|\Delta_{j+1/2}^n\right| .
\end{aligned}$$

Therefore we recover the following estimate for the total variation

$$\mathrm{TV}(\rho_{\Delta x}(T,\cdot)) \le (1+2w_\eta(0)\Delta t)^{T/\Delta t}\,\mathrm{TV}(\rho_{\Delta x}(0,\cdot)) \le e^{2w_\eta(0)T}\mathrm{TV}(\rho_0) .$$

From Proposition 2, the following space-time BV estimate can be derived as in [14, Corollary 5.1], relying on the Lipschitz continuity of g .

Corollary 1 *Let $\rho_0 \in BV(\mathbb{R};[0,1])$, and let $\rho_{\Delta x}$ be given by (15), (16). If $\alpha \ge 1+2\,\Delta x\, w_\eta(0)$ and $\Delta t \le 2\Delta x/(2\alpha+3\Delta x\, w_\eta(0))$, then for every $T>0$ there exists $\widetilde{C}=\widetilde{C}(w_\eta,\rho_0,T,\alpha)$ such that*

$$TV(\rho_{\Delta x};[0,T]\times\mathbb{R}) \le \widetilde{C}. \tag{24}$$

3.3 Discrete entropy inequalities

Following [3, Proposition 2.8], we derive a discrete entropy inequality for the approximate solution generate by (15), (16). Let us denote by

$$\begin{aligned}
G_{j+1/2}(u,v) &= \frac{1}{2}u\,V_j^n+\frac{1}{2}v\,V_{j+1}^n+\frac{\alpha}{2}(u-v) ,\\
F_{j+1/2}^\kappa(u,v) &= G_{j+1/2}(u\wedge\kappa, v\wedge\kappa)-G_{j+1/2}(u\vee\kappa, v\vee\kappa),
\end{aligned}$$

with $a\wedge b=\max(a,b)$ and $a\vee b=\min(a,b)$.

Proposition 3 *Let ρ_j^n, $j \in \mathbb{Z}$, $n \in \mathbb{N}$, be given by (14), (15). Then, if $\alpha \geq 1$ and the CFL condition (18) holds, for all $j \in \mathbb{Z}$, $n \in \mathbb{N}$ we have*

$$\left|\rho_j^{n+1} - \kappa\right| - \left|\rho_j^n - \kappa\right| + \lambda \left(F_{j+1/2}^{\kappa}(\rho_j^n, \rho_{j+1}^n) - F_{j-1/2}^{\kappa}(\rho_{j-1}^n, \rho_j^n)\right) + \frac{\lambda}{2} \operatorname{sgn}(\rho_j^{n+1} - \kappa)\, \kappa \left(V_{j+1}^n - V_{j-1}^n\right) \leq 0 \tag{25}$$

for all $\kappa \in \mathbb{R}$.

Proof Setting

$$\tilde{H}_j(u, v, z) = v - \lambda \left(G_{j+1/2}(v, z) - G_{j-1/2}(u, v)\right) ,$$

the function $\tilde{H}_j$ is monotone non-decreasing in its first variable, monotone non-decreasing in its second variable for $\alpha\lambda \leq 1$, which is guaranteed by the CFL condition (18), and monotone non-decreasing in its third variable for $\alpha \geq 1$, which is guaranteed by (19). Moreover, we have the identity

$$\begin{aligned}\tilde{H}_j(\rho_{j-1}^n \wedge \kappa, \rho_j^n \wedge \kappa, \rho_{j+1}^n \wedge \kappa) &- \tilde{H}_j(\rho_{j-1}^n \vee \kappa, \rho_j^n \vee \kappa, \rho_{j+1}^n \vee \kappa) \\ &= \left|\rho_j^n - \kappa\right| - \lambda \left(F_{j+1/2}^{\kappa}(\rho_j^n, \rho_{j+1}^n) - F_{j-1/2}^{\kappa}(\rho_{j-1}^n, \rho_j^n)\right) .\end{aligned}$$

By monotonicity,

$$\begin{aligned}
&\tilde{H}_j(\rho_{j-1}^n \wedge \kappa, \rho_j^n \wedge \kappa, \rho_{j+1}^n \wedge \kappa) - \tilde{H}_j(\rho_{j-1}^n \vee \kappa, \rho_j^n \vee \kappa, \rho_{j+1}^n \vee \kappa) \\
&= \tilde{H}_j(\rho_{j-1}^n, \rho_j^n, \rho_{j+1}^n) \wedge \tilde{H}_j(\kappa, \kappa, \kappa) - \tilde{H}_j(\rho_{j-1}^n, \rho_j^n, \rho_{j+1}^n) \vee \tilde{H}_j(\kappa, \kappa, \kappa) \\
&= \left|\tilde{H}_j(\rho_{j-1}^n, \rho_j^n, \rho_{j+1}^n) - \tilde{H}_j(\kappa, \kappa, \kappa)\right| \\
&= \operatorname{sgn}\left(\tilde{H}_j(\rho_{j-1}^n, \rho_j^n, \rho_{j+1}^n) - \tilde{H}_j(\kappa, \kappa, \kappa)\right) \times \left(\tilde{H}_j(\rho_{j-1}^n, \rho_j^n, \rho_{j+1}^n) - \tilde{H}_j(\kappa, \kappa, \kappa)\right) \\
&= \operatorname{sgn}\left(\tilde{H}_j(\rho_{j-1}^n, \rho_j^n, \rho_{j+1}^n) - \kappa + \frac{\lambda}{2}\kappa \left(V_{j+1}^n - V_{j-1}^n\right)\right) \\
&\quad \times \left(\tilde{H}_j(\rho_{j-1}^n, \rho_j^n, \rho_{j+1}^n) - \kappa + \frac{\lambda}{2}\kappa \left(V_{j+1}^n - V_{j-1}^n\right)\right) \\
&\geq \operatorname{sgn}\left(\tilde{H}_j(\rho_{j-1}^n, \rho_j^n, \rho_{j+1}^n) - \kappa\right) \times \left(\tilde{H}_j(\rho_{j-1}^n, \rho_j^n, \rho_{j+1}^n) - \kappa + \frac{\lambda}{2}\kappa \left(V_{j+1}^n - V_{j-1}^n\right)\right) \\
&= \left|\tilde{H}_j(\rho_{j-1}^n, \rho_j^n, \rho_{j+1}^n) - \kappa\right| + \frac{\lambda}{2} \operatorname{sgn}\left(\tilde{H}_j(\rho_{j-1}^n, \rho_j^n, \rho_{j+1}^n) - \kappa\right) \kappa \left(V_{j+1}^n - V_{j-1}^n\right) \\
&= \left|\rho_j^{n+1} - \kappa\right| + \frac{\lambda}{2} \operatorname{sgn}\left(\rho_j^{n+1} - \kappa\right) \kappa \left(V_{j+1}^n - V_{j-1}^n\right) ,
\end{aligned}$$

by definition of the scheme (15), (16), which gives (25).

3.4 $\mathbf{L^1}$ stability estimates

In this section, we prove explicit $\mathbf{L^1}$ estimates that ensure the stability of the scheme (15), (16).

Proposition 4 *Let $\rho_0, \bar\rho_0 \in BV(\mathbb{R};[0,1])$ be two initial data, and denote by $\rho_{\Delta x}, \bar\rho_{\Delta x}$ the corresponding approximate solutions constructed applying the modified Lax-Friedrichs scheme* (15), (16)*:*

$$\rho_j^{n+1} = \rho_j^n + \frac{\lambda\alpha}{2}\left(\rho_{j-1}^n - 2\rho_j^n + \rho_{j+1}^n\right) + \frac{\lambda}{2}\left(\rho_{j-1}^n V_{j-1}^n - \rho_{j+1} V_{j+1}^n\right) , \quad (26)$$

$$\bar\rho_j^{n+1} = \bar\rho_j^n + \frac{\lambda\alpha}{2}\left(\bar\rho_{j-1}^n - 2\bar\rho_j^n + \bar\rho_{j+1}^n\right) + \frac{\lambda}{2}\left(\bar\rho_{j-1}^n \bar V_{j-1}^n - \bar\rho_{j+1} \bar V_{j+1}^n\right) , \quad (27)$$

where we have set $V_j^n = 1 - \Delta x \sum_{k=0}^{N-1} w_\eta^k\, \rho_{j+k}^n$ and $\bar V_j^n = 1 - \Delta x \sum_{k=0}^{N-1} w_\eta^k\, \bar\rho_{j+k}^n$. Then, under assumptions (18) *and* (19)*, the following estimate holds:*

$$\left\|\rho_{\Delta x}(T,\cdot) - \bar\rho_{\Delta x}(T,\cdot)\right\|_{\mathbf{L^1}} \le K(w_\eta, \rho_0, \bar\rho_0, T)\left\|\rho_0 - \bar\rho_0\right\|_{\mathbf{L^1}} \quad (28)$$

with $K(w_\eta, \rho_0, \bar\rho_0, T) := \exp(T w_\eta(0)(1 + 0.5 \min\{C(w_\eta, \rho_0, T), C(w_\eta, \bar\rho_0, T)\}))$.

Proof Subtracting (27) from (26) we get

$$\begin{aligned}
\rho_j^{n+1} - \bar\rho_j^{n+1} = & \\
= & (1-\lambda\alpha)(\rho_j^n - \bar\rho_j^n) + \frac{\lambda\alpha}{2}(\rho_{j-1}^n - \bar\rho_{j-1}^n) + \frac{\lambda\alpha}{2}(\rho_{j+1}^n - \bar\rho_{j+1}^n) \\
& + \frac{\lambda}{2}\left(\rho_{j-1}^n (V_{j-1}^n - \bar V_{j-1}^n) + (\rho_{j-1}^n - \bar\rho_{j-1}^n)\bar V_{j-1}^n\right) \\
& - \frac{\lambda}{2}\left((\rho_{j+1}^n - \bar\rho_{j+1}^n)\bar V_{j+1}^n + \rho_{j+1}^n (V_{j+1}^n - \bar V_{j+1}^n)\right) \\
= & \left(1 - \lambda\alpha - \frac{\lambda}{2}\Delta x\, w_\eta^1\, \rho_{j-1}^n\right)(\rho_j^n - \bar\rho_j^n) \\
& + \frac{\lambda}{2}(\alpha + \bar V_{j-1}^n - \Delta x\, w_\eta^0\, \rho_{j-1}^n)(\rho_{j-1}^n - \bar\rho_{j-1}^n) \\
& + \frac{\lambda}{2}(\alpha - \bar V_{j+1}^n + \Delta x\, w_\eta^0\, \rho_{j+1}^n - \Delta x\, w_\eta^2\, \rho_{j-1}^n)(\rho_{j+1}^n - \bar\rho_{j+1}^n) \\
& + \frac{\lambda}{2}\Delta x \sum_{k=2}^{N-2}\left(w_\eta^{k-1}\rho_{j+1}^n - w_\eta^{k+1}\rho_{j-1}^n\right)(\rho_{j+k}^n - \bar\rho_{j+k}^n) \\
& + \frac{\lambda}{2}\Delta x\, w_\eta^{N-2}\, \rho_{j+1}^n(\rho_{j+N-1}^n - \bar\rho_{j+N-1}^n) + \frac{\lambda}{2}\Delta x\, w_\eta^{N-1}\, \rho_{j+1}^n(\rho_{j+N}^n - \bar\rho_{j+N}^n).
\end{aligned}$$

Observe that the coefficient of the first term is positive thanks to (18) and the coefficients of the second and third terms are positive thanks to (19). Therefore, taking the absolute values in the above equality we get

$$\left|\rho_j^{n+1} - \bar\rho_j^{n+1}\right| \le$$

$$
\begin{aligned}
&\leq \left(1 - \lambda\alpha - \frac{\lambda}{2}\Delta x\, w_\eta^1\, \rho_{j-1}^n\right) \left|\rho_j^n - \bar\rho_j^n\right| \\
&\quad + \frac{\lambda}{2}(\alpha + \bar V_{j-1}^n - \Delta x\, w_\eta^0\, \rho_{j-1}^n)\left|\rho_{j-1}^n - \bar\rho_{j-1}^n\right| \\
&\quad + \frac{\lambda}{2}(\alpha - \bar V_{j+1}^n + \Delta x\, w_\eta^0\, \rho_{j+1}^n - \Delta x\, w_\eta^2\, \rho_{j-1}^n)\left|\rho_{j+1}^n - \bar\rho_{j+1}^n\right| \\
&\quad + \frac{\lambda}{2}\Delta x \sum_{k=2}^{N-2} \left|w_\eta^{k-1}\rho_{j+1}^n - w_\eta^{k+1}\rho_{j-1}^n\right|\left|\rho_{j+k}^n - \bar\rho_{j+k}^n\right| \\
&\quad + \frac{\lambda}{2}\Delta x\, w_\eta^{N-2}\, \rho_{j+1}^n\left|\rho_{j+N-1}^n - \bar\rho_{j+N-1}^n\right| + \frac{\lambda}{2}\Delta x\, w_\eta^{N-1}\, \rho_{j+1}^n\left|\rho_{j+N}^n - \bar\rho_{j+N}^n\right| .
\end{aligned}
$$

Summing over $j \in \mathbb{Z}$, rearranging the indexes and observing that by monotonicity of w_η and using the triangular inequality

$$
\left|w_\eta^{k-1}\rho_{j+1}^n - w_\eta^{k+1}\rho_{j-1}^n\right| \leq (w_\eta^{k-1} - w_\eta^{k+1})\rho_{j+1}^n + w_\eta^{k+1}\left|\rho_{j+1}^n - \rho_{j-1}^n\right|
$$

we get

$$
\begin{aligned}
\sum_j \left|\rho_j^{n+1} - \bar\rho_j^{n+1}\right| \leq &\Bigg[1 + \frac{\Delta t}{2}\left(\sum_{k=2}^{N-2}(w_\eta^{k-1} - w_\eta^{k+1}) + w_\eta^{N-2} + w_\eta^{N-1}\right) \\
&\quad + \frac{\Delta t}{2}\sum_{k=2}^{N-2} w_\eta^{k+1}\left|\rho_{j-k+1}^n - \rho_{j-k-1}^n\right|\Bigg] \sum_j \left|\rho_j^n - \bar\rho_j^n\right| .
\end{aligned}
$$

Therefore,

$$
\|\rho_{\Delta x}(T,\cdot) - \bar\rho_{\Delta x}(T,\cdot)\|_{\mathbf{L^1}} \leq \left(1 + \Delta t\, \frac{2 + \mathrm{TV}(\rho_{\Delta x}(T))}{2} w_\eta(0))\right)^{T/\Delta t} \|\rho_0 - \bar\rho_0\|_{\mathbf{L^1}},
$$

which gives the desired estimate (28).

4 Proof of Theorem 1

We follow a Lax-Wendroff type argument, see [14, Theorem 5.3] and [19, Theorem 12.1], to show that the scheme (15), (16), converges to a weak solution of (1), in the sense of Definition 1. The main difficulty here is to deal with a numerical flux also depending on Δx (explicitly, and in the number of variables). Therefore, the classical argument relying on flux consistency and Lipschitz dependence must be replaced by direct estimates given by (31) below.

By Proposition 1 and Corollary 1, we can apply Helly's theorem, stating that there exists a subsequence, still denoted by $\rho_{\Delta x}$, that converges to some $\rho \in (\mathbf{L^1} \cap \mathbf{L^\infty} \cap \mathrm{BV})(\mathbb{R}^+ \times \mathbb{R}; [0,1])$ in the $\mathbf{L^1_{loc}}$-norm.

Let $\varphi \in \mathbf{C^1_c}(\mathbb{R}^2)$ and multiply (15) by $\varphi(t^n, x_j)$. Summing over $j \in \mathbb{Z}$ and $n \in \mathbb{N}$ we get

$$
\sum_n \sum_j \varphi(t^n, x_j)\left(\rho_j^{n+1} - \rho_j^n\right)
$$

$$= -\lambda \sum_n \sum_j \varphi(t^n, x_j) \left(g(\rho_j^n, \dots, \rho_{j+N}^n) - g(\rho_{j-1}^n, \dots, \rho_{j+N-1}^n) \right).$$

Summing by parts we obtain

$$\sum_j \varphi(0, x_j) \rho_j^0 + \sum_n \sum_j \left(\varphi(t^n, x_j) - \varphi(t^{n-1}, x_j) \right) \rho_j^n \qquad (29)$$
$$+ \lambda \sum_n \sum_j \left(\varphi(t^n, x_{j+1}) - \varphi(t^n, x_j) \right) g(\rho_j^n, \dots, \rho_{j+N}^n) = 0.$$

Then we multiply (29) by Δx getting

$$\Delta x \sum_j \varphi(0, x_j) \rho_j^0 + \Delta x \Delta t \sum_n \sum_j \frac{\varphi(t^n, x_j) - \varphi(t^{n-1}, x_j)}{\Delta t} \, \rho_j^n \qquad (30)$$
$$+ \Delta x \Delta t \sum_n \sum_j \frac{\varphi(t^n, x_{j+1}) - \varphi(t^n, x_j)}{\Delta x} \, g(\rho_j^n, \dots, \rho_{j+N}^n) = 0.$$

By strong $\mathbf{L^1_{loc}}$ convergence of $\rho_{\Delta x} \to \rho$, it is straightforward to see that the first two terms in (30) converge to

$$\int_{-\infty}^{+\infty} \rho_0(x) \varphi(0, x) \, dx + \int_0^{+\infty} \int_{-\infty}^{+\infty} \rho(t, x) \varphi_t(t, x) \, dx dt$$

as $\Delta x \searrow 0$. Concerning the last term, since $\rho_j^n \in [0, 1]$ and $\left| V_j^n \right| \leq 1$ for all $j \in \mathbb{Z}$, $n \in \mathbb{N}$, we observe that

$$\begin{aligned}
&\left| g(\rho_j^n, \dots, \rho_{j+N}^n) - \rho_j^n V_j^n \right| \\
&\leq \frac{\alpha}{2} \left| \rho_{j+1}^n - \rho_j^n \right| + \frac{1}{2} \left| \rho_{j+1}^n V_{j+1}^n - \rho_j^n V_j^n \right| \\
&\leq \frac{\alpha}{2} \left| \rho_{j+1}^n - \rho_j^n \right| + \frac{1}{2} \left| (\rho_{j+1}^n - \rho_j^n) V_{j+1}^n + \rho_j^n (V_{j+1}^n - V_j^n) \right| \\
&\leq \frac{1+\alpha}{2} \left| \rho_{j+1}^n - \rho_j^n \right| + \frac{1}{2} \rho_j^n \, \Delta x \sum_{k=0}^{N-1} w_\eta^k \left| \rho_{j+k+1}^n - \rho_{j+k}^n \right| \\
&\leq \frac{1+\alpha}{2} \left| \rho_{j+1}^n - \rho_j^n \right| + \frac{1}{2} \, w_\eta(0) \, \mathrm{TV}(\rho_{\Delta x}(t^n, \cdot)) \, \Delta x \\
&\leq \frac{1+\alpha}{2} \left| \rho_{j+1}^n - \rho_j^n \right| + C'(w_\eta, \rho_0, T) \, \Delta x \,, \qquad (31)
\end{aligned}$$

where we have set $C'(w_\eta, \rho_0, T) = w_\eta(0) C(w_\eta, \rho_0, T)/2$ for $T \geq t^n$. Therefore, the last term in (30) can be rewritten as

$$\begin{aligned}
&\Delta x \Delta t \sum_n \sum_j \frac{\varphi(t^n, x_{j+1}) - \varphi(t^n, x_j)}{\Delta x} \, g(\rho_j^n, \dots, \rho_{j+N}^n) \\
&= \; \Delta x \Delta t \sum_n \sum_j \frac{\varphi(t^n, x_{j+1}) - \varphi(t^n, x_j)}{\Delta x} \, \rho_j^n V_j^n
\end{aligned}$$

$$+\,\Delta x\Delta t\sum_n\sum_j \frac{\varphi(t^n,x_{j+1})-\varphi(t^n,x_j)}{\Delta x}\times\left(g(\rho_j^n,\ldots,\rho_{j+N}^n)-\rho_j^nV_j^n\right).$$

Again by $\mathbf{L^1_{loc}}$ convergence of $\rho_{\Delta x}\to\rho$ and boundedness of w_η, the first term in the above decomposition converges to

$$\int_0^{+\infty}\int_{-\infty}^{+\infty}\rho(t,x)v(\rho*_d w_\eta(t,x))\varphi_x(t,x)\;dxdt\;,$$

while the second term can be bounded using (31): set $T>0$ and $R>0$ such that $\varphi(t,x)=0$ for $t>T$ and $|x|>R$, and let $n_T\in\mathbb{N}$ and $j_0,j_1\in\mathbb{Z}$ such that $T\in\,]n_T\Delta t,(n_T+1)\Delta t]$, $-R\in\,]x_{j_0-1/2},x_{j_0+1/2}]$ and $R\in\,]x_{j_1-1/2},x_{j_1+1/2}]$, then

$$\begin{aligned}
&\Delta x\Delta t\sum_n\sum_j\frac{\varphi(t^n,x_{j+1})-\varphi(t^n,x_j)}{\Delta x}\;\left(g(\rho_j^n,\ldots,\rho_{j+N}^n)-\rho_j^nV_j^n\right)\\
&\le\;\Delta x\Delta t\|\varphi_x\|_\infty\sum_{n=0}^{n_T}\sum_{j=j_0}^{j_1}\left(\frac{1+\alpha}{2}\;\left|\rho_{j+1}^n-\rho_j^n\right|+C'(w_\eta,\rho_0,T)\;\Delta x\right)\\
&=\;\frac{1+\alpha}{2}\|\varphi_x\|_\infty\Delta x\Delta t\sum_{n=0}^{n_T}\sum_{j=j_0}^{j_1}\left|\rho_{j+1}^n-\rho_j^n\right|+\|\varphi_x\|_\infty C'(w_\eta,\rho_0,T)2RT\Delta x\\
&\le\;\frac{1+\alpha}{2}\|\varphi_x\|_\infty\int_0^T\int_{-R}^R|\rho_{\Delta x}(t,x+\Delta x)-\rho_{\Delta x}(t,x)|\;dxdt\\
&\quad+\|\varphi_x\|_\infty C'(w_\eta,\rho_0,T)2RT\Delta x\\
&\le\;\frac{1+\alpha}{2}\|\varphi_x\|_\infty C(w_\eta,\rho_0,T)\Delta x+\|\varphi_x\|_\infty C'(w_\eta,\rho_0,T)2RT\Delta x\;,
\end{aligned}$$

which clearly goes to zero when $\Delta x\searrow 0$.

Concerning the entropy condition, we proceed as above to show that (25) converges to (5). Multiplying (25) by $\Delta x\,\varphi(t^n,x_j)\ge 0$ and then summing by parts we get

$$\begin{aligned}
0\le\;&\Delta x\sum_j\varphi(0,x_j)\left|\rho_j^0-\kappa\right|+\Delta x\Delta t\sum_n\sum_j\frac{\varphi(t^n,x_j)-\varphi(t^{n-1},x_j)}{\Delta t}\;\left|\rho_j^n-\kappa\right|\\
&+\Delta x\Delta t\sum_n\sum_j\frac{\varphi(t^n,x_{j+1})-\varphi(t^n,x_j)}{\Delta x}\;F^\kappa_{j+1/2}(\rho_j^n,\rho_{j+1}^n)\\
&-\Delta x\Delta t\sum_n\sum_j\operatorname{sgn}\left(\rho_j^{n+1}-\kappa\right)\kappa\;\frac{V_{j+1}^n-V_{j-1}^n}{2\Delta x}\;\varphi(t^n,x_j)\;.
\end{aligned}$$

Following the same steps as above, the first three terms in the sum clearly converge to

$$\int_0^{+\infty}\int_{-\infty}^{+\infty}\left(|\rho(t,x)-\kappa|\varphi_t+|\rho(t,x)-\kappa|v(\rho*_d w_\eta(t,x))\;\varphi_x\right)dxdt$$

$$+\int_{-\infty}^{+\infty} |\rho_0(x) - \kappa| \varphi(0,x)\ dx$$

as $\Delta x \searrow 0$. The third term can be decomposed as

$$\begin{aligned}
&\sum_n \sum_j \operatorname{sgn}\left(\rho_j^{n+1} - \kappa\right) \kappa\ \frac{V_{j+1}^n - V_{j-1}^n}{2\Delta x}\ \varphi(t^n, x_j) \\
&= \left(\operatorname{sgn}\left(\rho_j^{n+1} - \kappa\right) - \operatorname{sgn}\left(\rho_j^n - \kappa\right)\right) \kappa\ \frac{V_{j+1}^n - V_{j-1}^n}{2\Delta x}\ \varphi(t^n, x_j) \qquad (32)\\
&+ \operatorname{sgn}\left(\rho_j^n - \kappa\right) \kappa\ \frac{V_{j+1}^n - V_{j-1}^n}{2\Delta x}\ \varphi(t^n, x_j)\ .
\end{aligned}$$

The first term in (32) can be controlled by $C(w_\eta, \rho_0)\Delta x$, and the second clearly converges to

$$-\int_0^{+\infty} \int_{-\infty}^{+\infty} \operatorname{sgn}(\rho(t,x) - \kappa)\, \kappa\, V_x(t,x)\ \varphi(t,x) dx dt\ ,$$

providing the entropy inequality (5).

5 Numerical tests

In this section, we perform numerical simulations to show evidence of some properties of equation (1). In particular, we compare solutions depending on the kernel choice and its support.

For the tests presented below, the space domain is given by the interval $[-1, 1]$, the space discretization mesh $\Delta x = 0.002$ and $\eta = 0.1$, where not specified otherwise. Absorbing conditions are imposed at the boundaries. More precisely, at the right boundary, we add $N = \eta/\Delta x$ ghost cells and define $\rho_j^n = \rho^n_{\frac{2}{\Delta x}}$ for every $j = \frac{2}{\Delta x} + 1, \ldots, \frac{2}{\Delta x} + N$, thus extending the solution constantly equal to the last value inside the domain. This choice is particularly suitable for computing the solutions of Riemann problems. At the left boundary, we just need add one ghost cell, as in classical problems.

5.1 Kernel support location

We aim at comparing the behavior of solutions of (1) with centered and upstream convolutions, obtained taking the velocity v as

$$1 - \Delta x \sum_{k=-N/2}^{N/2} w_\eta^k\ \rho_{j+k} \simeq v\Big(\int_{x-\eta/2}^{x+\eta/2} \rho(t,y) w_\eta(y-x)\ dy\Big)\ , \qquad (33)$$

which models drivers looking both ahead and behind, and

$$1 - \Delta x \sum_{k=-N+1}^{0} w_\eta^k\ \rho_{j+k} \simeq v\Big(\int_{x-\eta}^{x} \rho(t,y) w_\eta(y-x)\ dy\Big)\ , \qquad (34)$$

corresponding to drivers taking into account traffic densities behind them.

Figure 1 shows the density profiles at time $t = 0.5$ corresponding to the initial condition

$$\rho_0(x) = \begin{cases} 0.6 & \text{if } x < 0, \\ 0.2 & \text{if } x > 0. \end{cases} \tag{35}$$

The solution to the classical scalar conservation law

$$\partial_t \rho + \partial_x(\rho(1-\rho)) = 0 \tag{36}$$

would be a rarefaction wave, as displayed in Figure 1, (a).

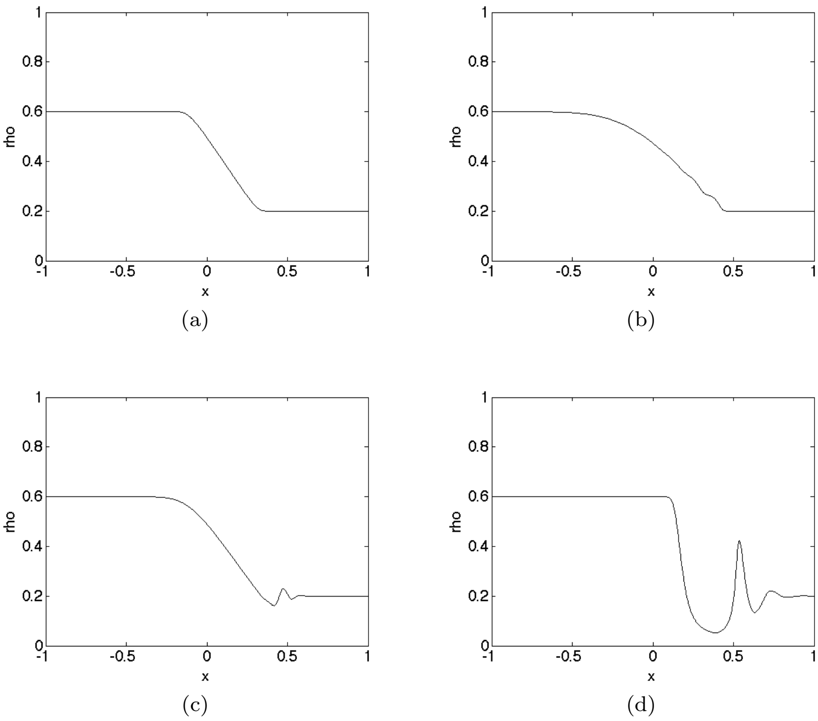

Fig. 1 Density profiles at $t = 0.5$ corresponding to a Riemann-like initial datum with $\rho_L = 0.6$, $\rho_R = 0.2$. Case (a) corresponds to the rarefaction profile that solves the classical equation (36). Cases (b), (c) and (d) correspond to the kernel $w_\eta(x) = 1/\eta$ with downstream (2), central (33) and upstream (34) supports respectively.

Figure 2 shows the density profiles at time $t = 0.2$ corresponding to the initial condition

$$\rho_0(x) = \begin{cases} 0.4 & \text{if } x < 0, \\ 0.9 & \text{if } x > 0. \end{cases} \tag{37}$$

The solution to the classical scalar conservation law would be a shock traveling with speed $\sigma = -0.3$.

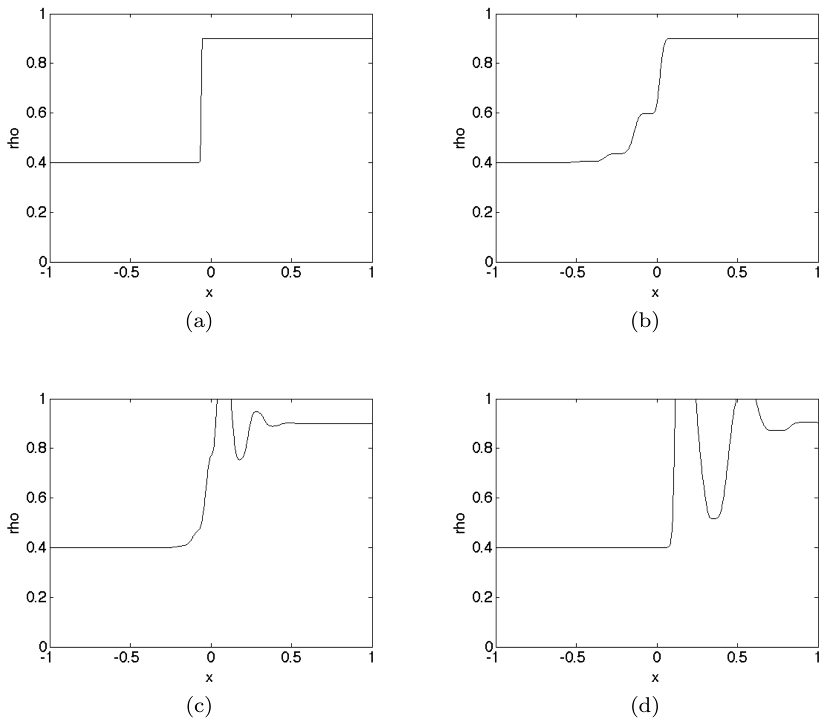

Fig. 2 Density profiles at $t = 0.2$ corresponding to a Riemann-like initial datum with $\rho_L = 0.4$, $\rho_R = 0.9$. Case (a) corresponds to the shock profile that solves the classical equation (36). Cases (b), (c) and (d) correspond to the kernel $w_\eta(x) = 1/\eta$ with downstream (2), central (33) and upstream (34) supports respectively.

Cases (b), (c) and (d) in Figures 1, 2 show the density profiles corresponding to the constant kernel $w_\eta(x) = 1/\eta$ with downstream (2), central (33) and upstream (34) supports respectively. (The approximate solutions do not change significantly using other non-increasing kernels.) We observe that the downstream convolution gives a smooth profile with constant total variation and monotonicity (as expected from Proposition 2) close to the solution of the corresponding classical problem (36) showed in Figures 1, 2, (a). On the contrary, central and upstream convolutions cause oscillations formation and total variation blow-up (for upstream convolution).

5.2 Kernel monotonicity

We are interested in studying the effect of the monotonicity of the kernel on the solution characteristics. To this aim, we consider the downstream convolution (2) and we compare the solution corresponding to the constant kernel $w_\eta(x) = 1/\eta$, to the linear decreasing kernel $w_\eta(x) = 2(\eta - x)/\eta^2$ and to the linear increasing kernel $w_\eta(x) = 2x/\eta^2$, for the Riemann-type initial datum

(37). Figure 3 shows the density profiles at time $t = 0.5$ (left) and the total variation $\mathrm{TV}(\rho(t,\cdot);[-1,1])$ for $t \in [0,1]$ (right) corresponding to the different kernel choices. Numerical simulations confirm that non-increasing monotonicity is necessary for the scheme to be monotonicity preserving. In particular, for Riemann type initial data, the total variation remains constant, while it increases for monotonically increasing kernels.

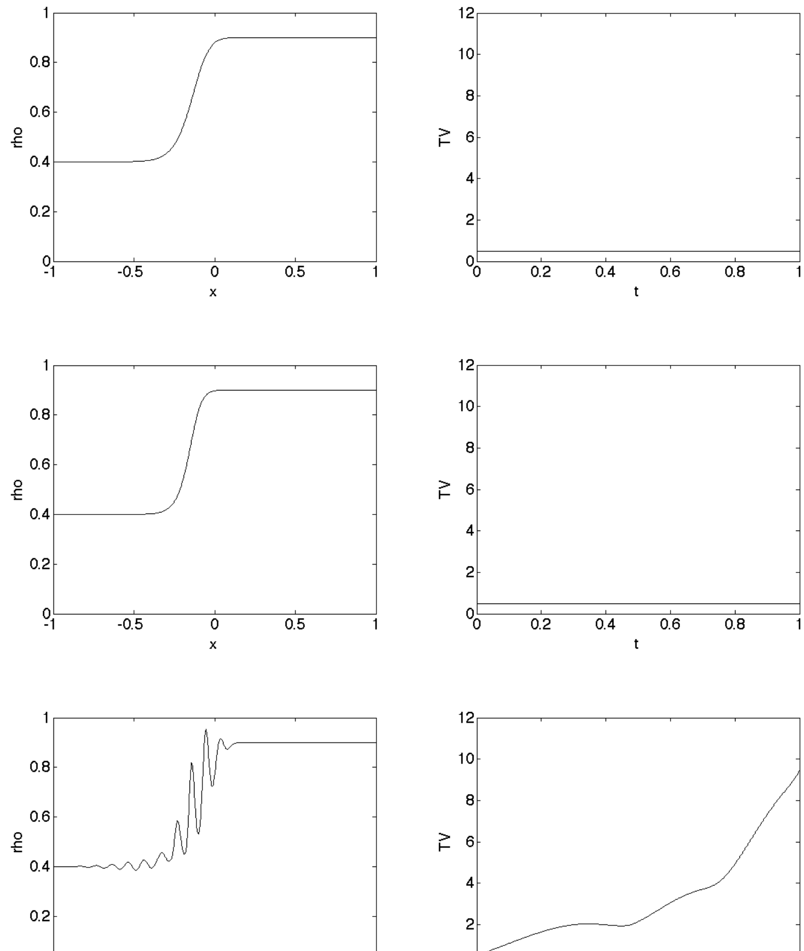

Fig. 3 Density profiles at $t = 0.5$ (left) and the total variation $\mathrm{TV}(\rho(t,\cdot);[-1,1])$ for $t \in [0,1]$ (right) corresponding to the Riemann-like initial datum with $\rho_L = 0.4$, $\rho_R = 0.9$ for $w_\eta(x) = 1/\eta$ (top), $w_\eta(x) = 2(\eta - x)/\eta^2$(middle) and $w_\eta(x) = 2x/\eta^2$ (bottom).

We also compute numerical convergence orders. Following [8], we define

$$\gamma(\Delta x) = \log_2\left(\frac{e(\Delta x)}{e(\Delta x/2)}\right), \tag{38}$$

where the $\mathbf{L^1}$-error is computed at final time $T = 0.3$ as

$$\begin{aligned} e(\Delta x) &= \left\|\rho_{\Delta x}(T,\cdot) - \rho_{\Delta x/2}(T,\cdot)\right\|_{\mathbf{L^1}} \\ &= \frac{\Delta x}{2}\sum_j \left|\rho_{\Delta x}(T,x_j) - \rho_{\Delta x/2}(T,x_j)\right|, \end{aligned} \tag{39}$$

where $x_j = (j-1/2)\Delta x/2$ for $\Delta x = 0.01, 0.005, 0.0025, 0.00125, 0.000625$. Table 1 show that non-increasing kernels give good convergence rates (the rate is far better for the linear decreasing kernel), while in the case of linear increasing kernel convergence is not clearly established. The $\mathbf{L^1}$-errors reported in the table represent the $\mathbf{L^1}$-distances to the reference solutions corresponding to $\Delta x = 0.00015625$.

	$w_\eta(x) = 1/\eta$		$w_\eta(x) = 2(\eta - x)/\eta^2$		$w_\eta(x) = 2x/\eta^2$	
Δx	$\gamma(\Delta x)$	$\mathbf{L^1}$-error	$\gamma(\Delta x)$	$\mathbf{L^1}$-error	$\gamma(\Delta x)$	$\mathbf{L^1}$-error
0.01	0.98021	3.013e-03	1.06427	3.315e-02	-0.42189	1.241e-01
0.005	0.93000	1.709e-03	1.06119	1.590e-02	-0.88509	1.287e-01
0.0025	0.61590	1.044e-03	0.87964	7.650e-03	-0.13054	1.303e-01
0.00125	0.44360	6.344e-04	1.05856	3.696e-03	0.15360	1.069e-01
0.000625	0.57113	3.632e-04	0.99995	1.547e-03	0.27699	7.093e-02

Table 1 Convergence orders (38) and $\mathbf{L^1}$-errors to the reference solutions corresponding to $\Delta x = 0.00015625$ for constant, linear decreasing and linear increasing kernels at final time $T = 0.5$ corresponding to a Riemann-like initial datum with $\rho_L = 0.4$, $\rho_R = 0.9$.

5.3 Limit $\eta \searrow 0$

In this section, we investigate the convergence to the solutions of the classical conservation law (36) as $\eta \searrow 0$. To this end, we compare the solutions to (1) with different values of $\eta = 0.1, 0.01, 0.001$, for $\Delta x = 0.001$ and initial datum (37). As expected, the solution computed with $\eta = 0.001$ coincides with the numerical solution of (1) computed using the Lax-Friedrichs type scheme

$$\rho_j^{n+1} = \rho_j^n + \frac{\lambda\alpha}{2}\left(\rho_{j-1}^n - 2\rho_j^n + \rho_{j+1}^n\right) + \frac{\lambda}{2}\left(f(\rho_{j-1}^n) - f(\rho_{j+1}^n)\right). \tag{40}$$

In fact, when $\eta = \Delta x$, $N = 1$ in (12) and the scheme (15), (16) coincides with (40).

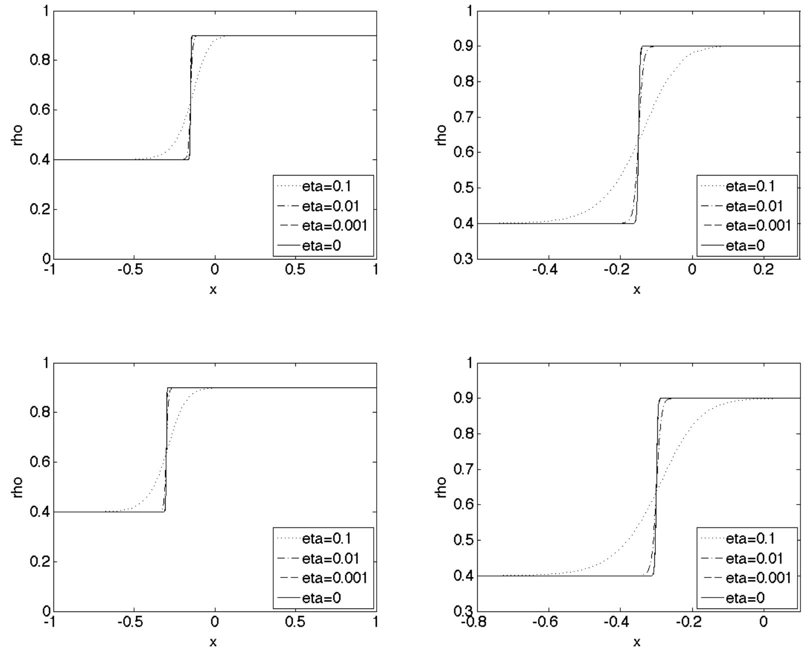

Fig. 4 Density profiles at $t = 0.5$ (top) and and $t = 1$ (bottom) corresponding to the Riemann-like initial datum with $\rho_L = 0.4$, $\rho_R = 0.9$ for $w_\eta(x) = 1/\eta$ and $\eta = 0.1, 0.01, 0.001$.

5.4 Highly oscillating initial datum

We consider the following initial datum

$$\rho_0(x) = \begin{cases} 0.5\,(1 + \sin(10\pi x)) & \text{if } x \in \ ]-0.5, 0.5[\,, \\ 0.5 & \text{otherwise,} \end{cases} \tag{41}$$

see Figure 5.

We are interested in exploring the effect of the location of the support of $w_\eta(x) = 1/\eta$ on the total variation of the solution. Figure 6 shows the density profiles at $t = 0.5$ (left) and the total variation $\mathrm{TV}(\rho(t,\cdot); [-1, 1])$ for $t \in [0, 0.5]$ (right). We observe that the downstream convolution (2) smooths down the solution towards the constant profile $\rho(t, x) = 0.5$ for $t \to \infty$. The central convolution (33) still reduces oscillations, but the upstream convolution induces blow up both in the $\mathbf{L}^\infty$ and in the BV norms.

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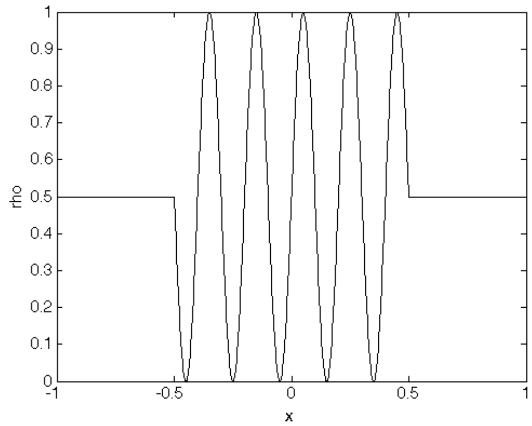

Fig. 5 Oscillating initial datum ρ_0 given by (41)

References

1. Amadori, D., Shen, W.: An integro-differential conservation law arising in a model of granular flow. J. Hyperbolic Differ. Equ. **9**(1), 105–131 (2012)
2. Amorim, P.: On a nonlocal hyperbolic conservation law arising from a gradient constraint problem. Bull. Braz. Math. Soc. (N.S.) **43**(4), 599–614 (2012)
3. Amorim, P., Colombo, R.M., Teixeira, A.: A numerical approach to scalar nonlocal conservation laws. ESAIM Math. Model. Numer. Anal. **49**(1), 19–37 (2015)
4. Aw, A., Rascle, M.: Resurrection of "second order" models of traffic flow. SIAM J. Appl. Math. **60**, 916–938 (2000)
5. Barth, T., Ohlberger, M.: Finite Volume Methods: Foundation and Analysis. John Wiley & Sons, Ltd (2004)
6. Betancourt, F., Bürger, R., Karlsen, K.H., Tory, E.M.: On nonlocal conservation laws modelling sedimentation. Nonlinearity **24**(3), 855–885 (2011)
7. Blandin, S., Work, D., Goatin, P., Piccoli, B., Bayen, A.: A general phase transition model for vehicular traffic. SIAM Journal on Applied Mathematics. **71**(1), 107–127 (2011)
8. Bretti, G., Natalini, R., Piccoli, B.: Numerical algorithms for simulations of a traffic model on road networks. J. Comput. Appl. Math. **210**(1-2), 71–77 (2007)
9. Colombo, R.M., Garavello, M., Lécureux-Mercier, M.: A class of nonlocal models for pedestrian traffic. Mathematical Models and Methods in Applied Sciences **22**(04), 1150,023 (2012)
10. Colombo, R.M., Herty, M., Mercier, M.: Control of the continuity equation with a non local flow. ESAIM Control Optim. Calc. Var. **17**(2), 353–379 (2011)
11. Colombo, R.M., Lécureux-Mercier, M.: Nonlocal crowd dynamics models for several populations. Acta Math. Sci. Ser. B Engl. Ed. **32**(1), 177–196 (2012)
12. Crippa, G., Lécureux-Mercier, M.: Existence and uniqueness of measure solutions for a system of continuity equations with non-local flow. NoDEA Nonlinear Differential Equations Appl. **20**(3), 523–537 (2013)
13. Dafermos, C.M.: Solutions in L^∞ for a conservation law with memory. In: Analyse mathématique et applications, pp. 117–128. Gauthier-Villars, Montrouge (1988)
14. Eymard, R., Gallouët, T., Herbin, R.: Finite volume methods. In: Handbook of numerical analysis, Vol. VII, Handb. Numer. Anal., VII, pp. 713–1020. North-Holland, Amsterdam (2000)
15. Harten, A.: High resolution schemes for hyperbolic conservation laws. J. Comput. Phys. **49**(3), 357–393 (1983)

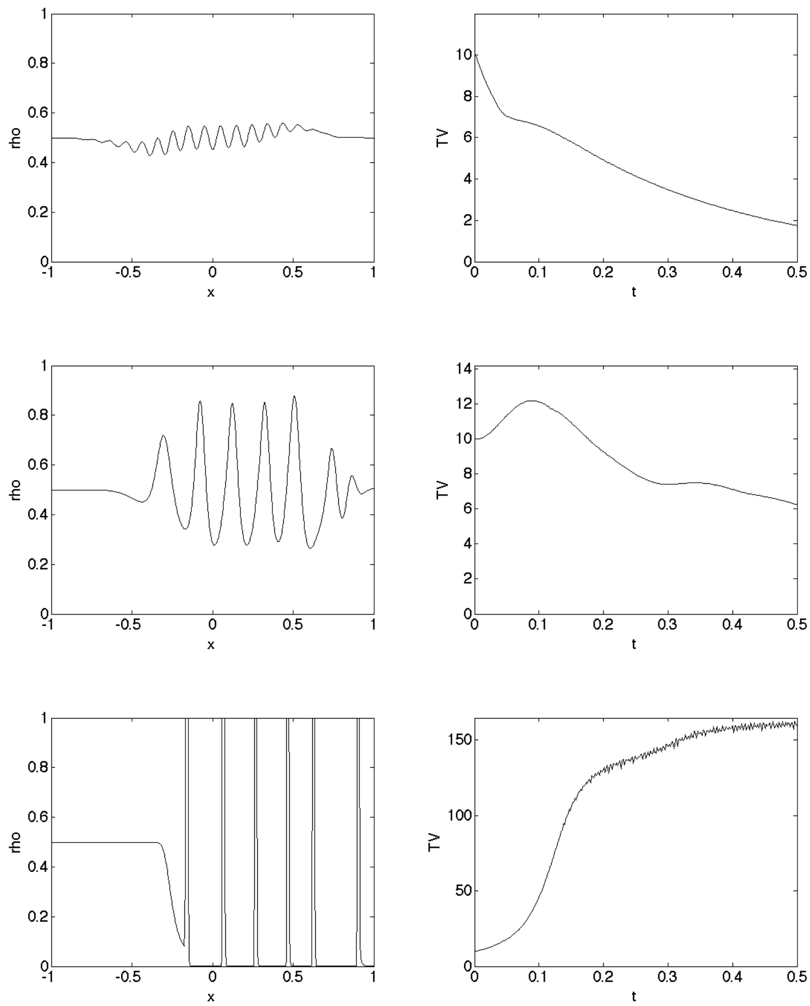

Fig. 6 Density profiles at $t = 0.5$ (left) and the total variation $\mathrm{TV}(\rho(t,\cdot);[-1,1])$ for $t \in [0,0.5]$ (right) corresponding to the initial datum (41) for $w_\eta(x) = 1/\eta$ with downstream (top), central (middle) and upstream (bottom) supports.

16. Karlsen, K.H., Risebro, N.H.: On the uniqueness and stability of entropy solutions of nonlinear degenerate parabolic equations with rough coefficients. Discrete Contin. Dyn. Syst. **9**(5), 1081–1104 (2003)
17. Kružkov, S.N.: First order quasilinear equations with several independent variables. Mat. Sb. (N.S.) **81 (123)**, 228–255 (1970)
18. Lebacque, J.P., Mammar, S., Salem, H.H.: Generic second order traffic flow modelling. In: International Symposium on Transportation and Traffic Theory 2007 (2007)
19. LeVeque, R.J.: Numerical methods for conservation laws, second edn. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel (1992)
20. Lighthill, M.J., Whitham, G.B.: On kinematic waves. II. A theory of traffic flow on long crowded roads. Proc. Roy. Soc. London. Ser. A. **229**, 317–345 (1955)
21. Richards, P.I.: Shock waves on the highway. Operations Res. **4**, 42–51 (1956)

22. Sopasakis, A., Katsoulakis, M.A.: Stochastic modeling and simulation of traffic flow: asymmetric single exclusion process with Arrhenius look-ahead dynamics. SIAM J. Appl. Math. **66**(3), 921–944 (electronic) (2006)