Sparsely observed agent-based systems: a generative model for instantaneous crowd modeling

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ABSTRACT

Real-time estimation of dynamical phenomena involving large number of agents is critical for Smarter Cities applications. However, for systems with many degrees of freedom, including crowds, the availability of timely structured and usable data remains a challenge. Given the high-dimensionality of the state-space, crowd systems are not fully observable for most applications. In this work, we consider the problem of model design for crowd estimation in a one dimensional space, given limitations on data stream availability. Our focus is on the model performance for estimation purposes, rather than comprehensive modeling capabilities. Starting from a classical cellular automaton model, we design a provably equivalent analytical instantaneous model, and we highlight modeling discrepancies between the modeling approaches.

Categories and Subject Descriptors

H.4 [**Information Systems Applications**]: Miscellaneous

General Terms

Modeling for agent based simulation

Keywords

Modeling

1. INTRODUCTION

Crowds modeling has many applications, from infrastructure planning to real-time monitoring, safety and security. A distinct feature of crowds is that agents have a high number of degrees of freedom (compared to vehicles for instance) and the agent decision process is not well captured by physical laws (compared to molecules for instance). Additionally, data availability has historically been a challenge for the crowd modeling community.

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1.1 Corridor problem

In this work, we illustrate the complexity of crowd modeling on the case of the simplest meaningful example, taken to be a one dimensional corridor of length L, see Figure 1. Pedestrians travel from the left to the right. The left end of the corridor (entrance) is always open, whereas the right end of the corridor (exit) can be open or closed.

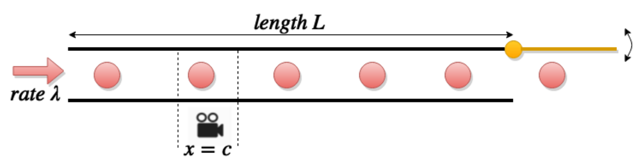

Figure 1: Corridor problem.

Time and space are assumed to be discrete, specifically, we denote x_n^i the number of pedestrians in cell i, at time n, where suitable discretization intervals Δx for space and Δt for time have been defined.

A natural linear model characterizing the evolution of such a system reads

$$x_n = A_n\, x_{n-1} + B_n\, u_n \qquad (1)$$

where $x_n = (x_n^1, \cdots, x_n^I)$ is the vector of pedestrian counts over the I cells, the matrix A_n characterizes the dynamics of pedestrians in the system at the previous time step $n-1$ (including exits), the scalar u_n denotes the number of pedestrians entering the system, and the matrix B_n characterizes the 1-step dynamics of pedestrians who just entered the system.

For practical applications, the problem of modeling the movement of the crowds is tightly related to available data. Indeed, it makes common sense that modeling complexity should be added to the system model only to the extent that the benefit of the enhancement can be justified and validated with calibration data.

In this work, we investigate error propagations for *natural* cellular automata model of crowds, in particular given realistic data availability, and we introduce a generative model more appropriate for crowd modeling given the relative sparsity of validation data points.

For simplicity we omit the case of bottlenecks or congestion created either by the infrastructure or by the crowd itself, i.e. we restrict our analysis to the case of the linear system (1).

1.2 Related Work

Agent-based models of crowds historically date back to the social force model [5], which proposed modeling mathematically the forces acting on individuals, and simulating the full set of interactions, or deriving macroscopic laws on aggregate quantities such as the flow of pedestrians, see for example [1] [10].

Crowds exhibit complex self-organizing behaviors [6] which also motivated the consideration of *multi-scale* approaches [2]. A typical challenge arising with crowds is the collection of validation and calibration data [7].

A significant amount of work in the estimation community has been focused on deriving estimation error for linear models with Gaussian statistics [8]. The value of interest evolves according to an imperfect dynamical model and is corrected at each step time with observations. This framework has been extended to the case of non-linear systems with some success, see [3], [4] and [9].

1.3 Contributions

In this work we introduce a new generative model able to capture the properties of available data for crowds. We propose an analytical framework to derive estimation error and we apply it to the comparison of propagation models and instantaneous models of crowds.

The rest of the article is organized as follows: Section 2 introduces the two model classes considered, in particular the novel generative instantaneous model, and presents equivalence and calibration results. In Section 3 we outline an estimation framework for the model parameters. Section 4 presents results on error propagation, assuming specific sensor configurations, and we provide results on the optimal sensor placement problem, for the two modeling frameworks. Finally, Section 5 presents concluding remarks and comments.

2. MATHEMATICAL MODELING

In this section we remind the properties of a *natural* propagation model for crowds in a corridor, and we then introduce a new generative model for such crowds.

2.1 Propagation model

A typical approach in dynamical system modeling consists of representing the theoretical physical understanding of the system in mathematical form. Let us note a_{ij} the coefficients of the matrix A (here we omit the time index for convenience) from equation (1). The coefficient a_{ij} characterizing the proportion of pedestrians moving from cell j to cell i within a time step, hence for the corridor problem, common sense suggests the following definition:

$$\begin{cases} \forall i, j \;\; a_{ij} \in [0,1] \\ \forall j \;\; \sum\limits_{i} a_{ij} \leq 1 \end{cases}$$

which models the fact that there is no destruction or creation of pedestrians within the system, and that some pedestrians may leave the system.

For the matrix B representing the 1 time-step evolution of pedestrians who just entered the system, the coefficients $b_{i,1}$ denoting the proportion of entering pedestrians reaching cell i, similar conservation properties are desirable

$$\begin{cases} \forall i \;\; b_{i,1} \in [0,1] \\ \sum b_{i,1} = 1 \end{cases} \tag{2}$$

which models the fact that the full amount amount u of exogenous passengers (and only that amount) can enter the corridor.

In order to further simplify our analysis, we also restrict the dynamical model considered to the case where all pedestrians eventually leave the system, namely that the coefficients $a_{i,j}$ satisfy

$$\exists N, \forall n > N, \mathbf{u}_n = 0 \Rightarrow \exists N' \;\; \forall n > N', \;\; \mathbf{x}_n = 0. \tag{3}$$

To simplify our analysis, we consider the following sufficient condition, which expresses that the matrix A must be nilpotent, and models the fact that at each time step, all pedestrians move towards the right.

PROPOSITION 1. *A sufficient condition on the coefficients* $a_{i,j}$ *to verify* (3) *is:*

$$\begin{cases} \forall i, j, \;\; i \leq j \Rightarrow a_{ij} = 0 \end{cases} \tag{4}$$

The dynamics of the passenger in the propagation model under condition (4) can be further illustrated as a tree, see Figure 2. Pedestrians can only enter at the roof (left end of the corridor) and can only travel towards the right (leaves of the tree).

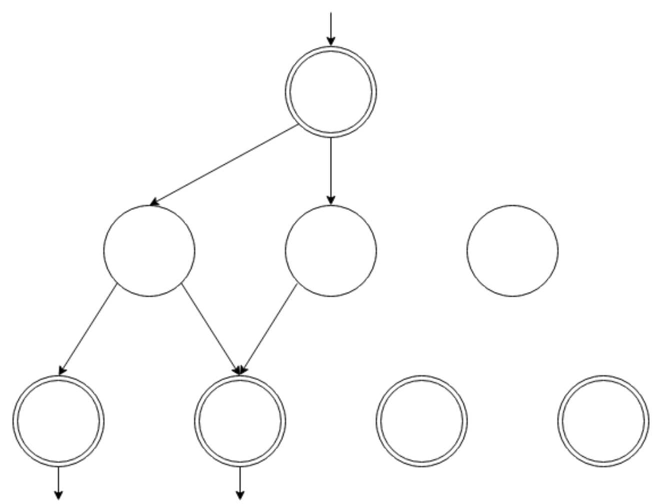

Figure 2: Tree of pedestrian dynamics in the corridor under the propagation model (nodes of the same depth correspond to the same physical cell).

Example.

Let us consider the case where the corridor exit can be open or closed, and assume that all pedestrians inside the corridor move move by exactly one cell at each time step. In particular pedestrians can only exit from the right-most cell, and pedestrians entering the corridor only reach the left-most cell. Hence the parameters of system (1) read:

$$\begin{cases} A^n = \begin{pmatrix} 0 & . & . & . & . & 0 \\ 1 & 0 & . & . & . & . \\ 0 & 1 & . & . & . & . \\ . & 0 & . & . & . & . \\ . & . & . & . & 0 & 0 \\ 0 & 0 & . & 0 & 1 & \delta_n \end{pmatrix} \\ B^n = \begin{pmatrix} 1 & 0 & .. & 0 \end{pmatrix}^T \end{cases} \tag{5}$$

where δ is a Dirac measure characterizing the opening of the corridor exit. It is clear that the true matrix coefficients must be estimated from available data.

In the following section we present a novel generative model in which we attempt to capture not the physical properties of the system, but the physical properties of observable variables, given realistic data.

2.2 Instantaneous model

The propagation model presented in the previous section requires knowing a number of parameters which is quadratic in the discretization of the state space. Namely the propagation model requires dynamical knowledge of point-to-point propagation speed for each location pair within the space considered.

While simplifications such as the unidirectional constraint presented in (4) make sense for the case of the corridor problem considered, it is clear that such assumptions fail in more complex environments.

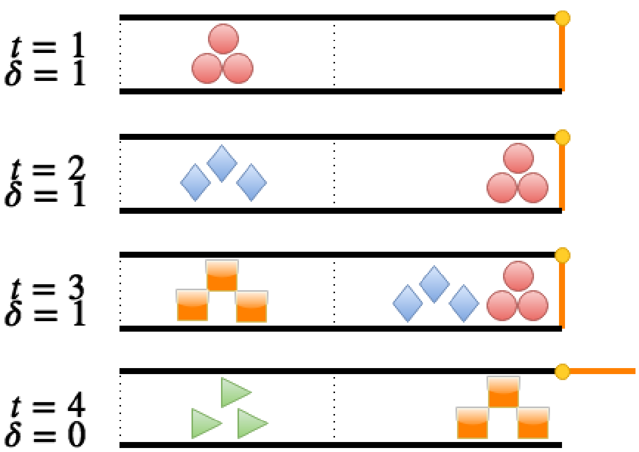

Figure 3: Propagation model: passengers move from left to right, passengers represented by the same icon entered the corridor at the same time. The corridor exit on the right-hand side can be open or closed.

The main source of data for quantitative observation of crowds and calibration of pedestrian models remains images, either recorded as images or extracted from a video file. In this section we introduce a novel generative model in which we attempt to model the crowd dynamics as observed from an image dataset, as a *sequence of static snapshots.*

Within the framework of system (1), we require the following properties for the matrix A of the instantaneous model:

$$\begin{cases} \forall i,j \;\; i \neq j \Rightarrow a_{ij} = 0 \\ \forall i,j \;\; a_{ij} \in [0,1] \end{cases} \tag{6}$$

which expresses the fact that the matrix A is diagonal (first equation), i.e. the movement of pedestrians is not modeled and pedestrians do not move inside the corridor, and that pedestrians already inside the system can only remain inside the system or exit.

We further require the following properties of the matrix B characterizing the behavior of entering pedestrians:

$$\begin{cases} \forall i \;\; b_{i,1} \in [0,1] \\ \sum b_{i,1} = 1 \end{cases} \tag{7}$$

which is identical to the requirement of the equivalent matrix for the propagation model (2). From a modeling perspective, similar to the propagation model, pedestrians are essentially dropping from the corridor ceiling, however in the instantaneous model pedestrians do not move and slowly fade from the place they landed at.

Illustrations of the qualitative behavior of both models are provided in Figure 3 and Figure 4.

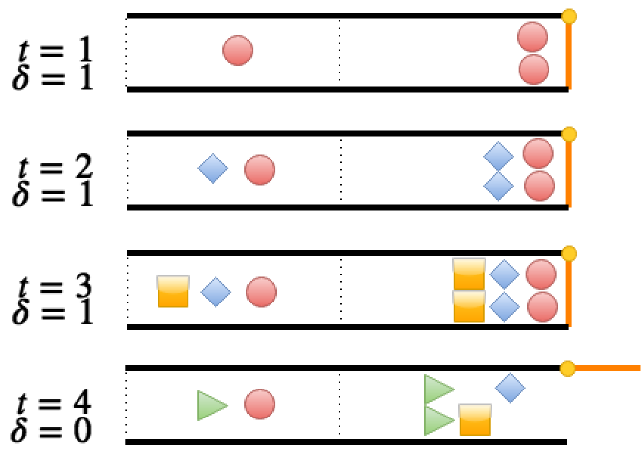

Figure 4: Instantaneous model: passengers do not move, only appear at a location and fade from the same location. Passengers represented by the same icon entered the corridor at the same time. The corridor exit on the right-hand side can be open or closed.

Example.

Let us consider the case where the corridor exit can be open or close, assuming that a proportion α of pedestrians exit from the corridor when the exit is open, and that a typical crowding of 2/3 of the crowd is observed in the last cell, the parameters of system (1) read:

$$\begin{cases} A = (\mathbb{1}(\delta = 1) + (1-\alpha)\mathbb{1}(\delta = 0))I_d \\ B = \begin{pmatrix} \frac{1}{(3\,(I-1))} & \frac{1}{(3\,(I-1))} & \cdots & \frac{2}{3} \end{pmatrix}^T \end{cases} \tag{8}$$

where δ is a Dirac measure characterizing the closing of the corridor exit, and we recall that I denotes the number of cells in the corridor. While it is clear that some parameters must still be estimated, one can observe that given the imposed model limitation (6) (7), the number of parameters is much reduced compared to the propagation model. We will formally show the impact of this difference on the calibration procedure subsequently.

In the following section we present an equivalence result between the two approaches, which illustrates that while the instantaneous approach is much simpler, on average its modeling properties are similar.

2.3 Modeling equivalence

In this section we prove an equivalence result for the simplified propagation model (5) and the instantaneous model (8). Let us assume that the corridor exit is open every r time units, namely that

$$\delta(t) = 0 \Leftrightarrow n \equiv 0 \; mod(r),$$

and let us further assume that the corridor entries have Poisson distribution of parameter λ. Then, the crowd levels for the two models are on average equal.

PROPOSITION 2. *Given the instantaneous model* (5) *with* $A = I_d$, *if the coefficients* α *and the matrix* B *are chosen such that:*

$$\begin{cases} b_{i,1} = \dfrac{1}{(I-1) + \frac{r+1}{2}} \; \textit{for } i < I \\ b_{I,1} = \dfrac{1}{1 + \frac{2(I-1)}{r+1}} \\ \alpha = \dfrac{1}{1 + \frac{I-1}{r}} \end{cases}$$

Then:

$$\forall i, \;\; \mathbb{E}(x^i_{prop} - x^i_{inst}) = 0$$

PROOF. For the propagation model with unit speed (5):

$$\forall n, \forall i < I, \; x^i_n = u_{n-i+1}$$

hence

$$\forall i < I, \; \mathbb{E}(x^i) = \lambda.$$

Besides, given that the exit only opens every r time units:

$$\forall n \equiv n' \; mod(r), \; x^I_n = \sum_{k=0}^{n'} u_{n-I+1-k}.$$

Thus:

$$\begin{aligned} \mathbb{E}(x^I_n) &= \sum_{k=0}^{n'} \mathbb{E}(u_{n-I+1-k}) \\ \mathbb{E}(x^I) &= \frac{1}{r}\sum_{n'=0}^{r-1}\sum_{k=0}^{n'}(n'+1)\lambda \\ &= \frac{r+1}{2}\lambda \end{aligned}$$

On the other hand, for the instantaneous model (8):

$$\forall i, \forall n \equiv n' \; mod(r),$$

$$x^i_n = b_{i,1}[\sum_{k=0}^{n'} u_{n-k} + \sum_{p=1}^{\infty}(1-\alpha)^p \sum_{k=0}^{r-1} u_{n-pr-k}],$$

thus

$$\mathbb{E}(x^i_n) = b_{i,1}\lambda[(n'+1) + r\frac{1-\alpha}{\alpha}],$$

and

$$\begin{aligned} \mathbb{E}(x^i) &= b_{i,1}\lambda[\frac{1}{r}\sum_{n'=0}^{r-1}[(n'+1) + r\frac{1-\alpha}{\alpha}]] \\ &= b_{i,1}\lambda[\frac{r+1}{2} + r\frac{1-\alpha}{\alpha}]. \end{aligned}$$

The equivalence condition thus reads:

$$\begin{cases} 1 = b_{i,1}[\frac{r+1}{2} + r\frac{1-\alpha}{\alpha}] \text{ for } i < I \\ \frac{r+1}{2} = b_{I,1}[\frac{r+1}{2} + r\frac{1-\alpha}{\alpha}] \\ \sum_{i=1}^{I} b_{i,1} = 1 \end{cases}$$

which leads to the following solution

$$\begin{cases} b_{i,1} = \dfrac{1}{(I-1) + \frac{r+1}{2}} \text{ for } i < I \\ b_{I,1} = \dfrac{1}{1 + \frac{2(I-1)}{r+1}} \\ \alpha = \dfrac{1}{1 + \frac{I-1}{r}} \end{cases}$$

□

REMARK 1. *The value of* α *identified by Proposition 2 corresponds to the average proportion of pedestrians in cell* $i = I$ *in the propagation model. Similarly* $b_{I,1} = \frac{r+1}{2} b_{i<I,1}$ *because in the propagation model pedestrians spend in average* $\frac{r+1}{2}$ *more time in cell* $i = I$ *than in the other cells.*

2.4 Calibration properties

In this section we consider the influence of parameters calibration on the estimation error for the two modeling framework. For simplicity, we assume that the corridor exit is always open. We assume that the *true model* is a propagation model satisfying the dynamics (1), with matrix A and B. We denote all parameters or estimated quantities from the estimation model (propagation or instantaneous) as $\widehat{x}$, while regular notation x denote the true parameters or true quantities.

Propagation model.

For the propagation model the parameters to estimate are the coefficients of the propagation matrix $\widehat{A}$. Let us assume that $\widehat{a}_{i+1,i} = a_{i+1,i} + \epsilon_i$ where ϵ_i is a Gaussian random variable with zero mean and variance σ^2.

PROPOSITION 3. *Under the previous assumptions, the total error for the propagation model grows asymptotically as the square of the number of cells* $\mathcal{O}(I^2)$.

PROOF.

$$\begin{aligned} \mathbb{E}||\widehat{x}^j_n - x^j_n||^2 &= \mathbb{E}||\prod_{i=1}^{j} a_{i+1,i} - \prod_{i=1}^{j}(a_{i+1,i} + \epsilon_i)||^2 ||u^{n-j+1}||^2 \\ &\approx j\sigma^2 ||u^{n-j+1}||^2 \end{aligned}$$

Thus the sum of the errors for the entire corridor grows as $\mathcal{O}(I^2)$. □

Instantaneous model.

For the instantaneous model, the parameters to estimate are the $b_{i,1}$. Let us assume that $\widehat{b}_{i,1} = b_{i,1} + \epsilon_i$ where ϵ_i is a Gaussian random variable with zero mean and variance σ^2.

PROPOSITION 4. *Under the previous assumptions, the total error for the instantaneous model grows asymptotically as a linear function of the number of cells $\mathcal{O}(I)$.*

PROOF. By construction $\widehat{x}_n^j \propto \widehat{b}_{i,1}$ and does not depend on the other parameters. Thus the local error is proportional to σ^2 and the total error is a $\mathcal{O}(I)$. □

We observe that the errors tend to increase faster in the propagation model because there is an inner dependency between parameters, which causes errors to not only propagate but create comparable errors. Comparatively, parameter errors in the instantaneous model have no impact on other errors.

3. ESTIMATION FRAMEWORK

In this section we present a framework for real-time estimation using the propagation model framework or the instantaneous model framework. We first recall a fundamental result of estimation, which supports the subsequent analysis.

LEMMA 1. *Given two estimators $\widehat{X}_1$ and $\widehat{X}_2$ of X such that:*

$$\begin{cases} \widehat{X}_1 = X + W_1 \\ \widehat{X}_2 = X + W_2 \\ W_1 \sim \mathcal{N}(0,\sigma_1),\, W_2 \sim \mathcal{N}(0,\sigma_2) \end{cases}$$

The linear combination $X^* = \alpha\widehat{X}_1 + (1-\alpha)\widehat{X}_2$ that minimizes $\mathbb{E}[(X - X^*)^2]$ is:

$$X^* = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}\widehat{X}_1 + \frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}\widehat{X}_2$$

with variance:

$$\sigma^{*2} = \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}$$

3.1 Bayesian network formulation

We consider our dynamical system (1) with online observations from sensors.

$$\begin{cases} \mathbf{x}_n = A_n\mathbf{x}_{n-1} + B_n\mathbf{u}_t + \mathbf{w} \\ \\ \mathbf{y}_n = H_n\mathbf{x}_n + \mathbf{v} \end{cases} \tag{9}$$

where $\mathbf{x}_n$ and $\mathbf{y}_n$ are respectively the state of the system and its observation at time n, $\mathbf{u}_n$ is exogenous and $\mathbf{w}$ and $\mathbf{v}$ are error terms. Conditional dependencies between variables are expressed in Figure 5.

The mode δ of the corridor exit determines both the entries of pedestrians from the right, and the inner evolution of the system (passengers can exit only if the exit is open). Equation (9) expresses that, given the state estimate, the observation is independent from all other variables and the state estimate at time $n+1$ is independent from all the entries and modes before n.

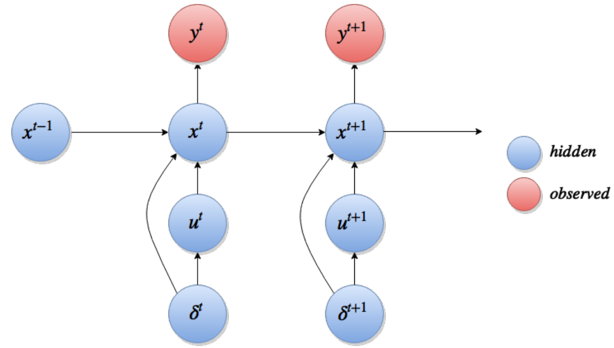

Figure 5: Graphical model: for the corridor problem with two-way pedestrian flow.

3.2 Noise modeling

Our model involves two types of errors, the error on the evolution and the error on the observations. In this work, we focus our design on the model error. The error on the evolution corresponds to the uncertainty that we have on the system dynamics. We also make the general assumption that it is centered Gaussian,

$$\mathbf{w} \sim \mathcal{N}(0, Q).$$

3.2.1 *Propagation model*

For the propagation model, we distinguish entry cells from other cells. The error on the entry cells corresponds to error on pedestrian inflow, while the error on inner cells corresponds to inner dynamics error compounded with inflow error. Thus we define the covariance matrix Q as

$$\tilde{Q} = diag(\boldsymbol{\sigma}^2)$$

where:

$$\sigma_i^2 = \begin{cases} \sigma_u^2 \text{ if } i \text{ is an entry cell} \\ \sigma_W^2 \text{ otherwise} \end{cases}$$

3.2.2 *Instantaneous model*

For the instantaneous model, the dynamics is such that the errors on distinct cells are not independent. We propose that the correlation error depends on two properties

- the physical distance between two cells: $||\mathbf{v}_i - \mathbf{v}_j||$,
- the historical distance between two cells: $|b_{i,1} - b_{j,1}|$,

expressing the fact that given the model dynamics, we expect error between close cells to be similar, and errors between cells with similar flow to be similar. This is motivated by the fact that the pedestrian density is expected to be relatively smooth spatially. Thus we propose to set

$$\begin{cases} \tilde{Q}_{ii} = \sigma_i^2 \\ \tilde{Q}_{ij} = \sigma_i\sigma_j e^{-min(\frac{1}{Z_1}||\mathbf{v}_i - \mathbf{v}_j||, \frac{1}{Z_2}|b_{i,1}-b_{j,1}|)} \end{cases}$$

REMARK 2. *Although this matrix is always symmetric, there is no guarantee in practice that it is positive semi-definite.*

3.3 Mode estimation

In practice, the corridor exit status is not perfectly known and must be estimated.

When we do not have exogenous estimates, our choice is driven by the likelihood of observing the sequence $\{\mathbf{y}_n, \mathbf{y}_{n-1}, .., \mathbf{y}_1\}$ depending on the value of δ. We want to choose the value of δ that maximizes the likelihood: $\delta \mapsto \mathbb{P}(\mathbf{y}|\delta)$.

In a static approach, $\mathbb{P}(\delta = 0)$ is simply the frequency of switch between the two modes. This probability is equal to $\frac{1}{r}$ in the corridor case.

In the propagation model, the only state variable of $\mathbf{x}$ that depends on the mode of the system is the pedestrian crowd in the right-most cell. So we need to assume that this cell is observed to build a model. As previously mentioned, the number of pedestrians in cell I is periodic with period r and as pedestrians accumulate when the mode is closed:

$$\mathbf{E}(x_0^I) < \mathbf{E}(x_1^I) < .. < \mathbf{E}(x_{r-1}^I)$$

The opening of the corridor exit corresponds to $n \equiv 0 \, mod(r)$. So we can choose:

$$\eta(y) = \begin{cases} 1 \text{ if } y > y^* \\ 0 \text{ if } y \leq y^* \end{cases}$$

where, y^* is such that:

$$\mathbb{P}(x_0^I = y^*) = \mathbb{P}(x_1^I = y^*)$$

Proposition 5. *For the propagation model, the observation threshold reads*

$$y^* = \frac{\lambda}{ln(2)}$$

Proof. If $n \equiv i \, mod(r)$, x_n^I is the sum of the realizations of $i+1$ Poisson variables of parameter λ. Thus we look for k such that:

$$\frac{\lambda^k}{k!}e^{-\lambda} = \sum_{i=0}^{k}\frac{\lambda^i}{i!}e^{-\lambda}\frac{\lambda^{k-i}}{(k-1)!}e^{-\lambda} \Leftrightarrow \frac{1}{k!} = e^{-\lambda}\sum_{i=0}^{k}\frac{\lambda^i}{i!(k-i)!}$$

$$\Leftrightarrow e^{\lambda} = \sum_{i=0}^{k} C_k^i$$

$$\Leftrightarrow e^{\lambda} = 2^k$$

$$\Leftrightarrow k = \frac{\lambda}{ln(2)}$$

□

For the instantaneous model we can adapt the same mechanism but instead of looking only for the critical value, we can also attempt identifying the drop in the number of pedestrians that is a consequence of the opening of the door:

$$\eta_1(y_n) = \mathbf{1}(y_n > y^*)$$

$$\eta_2(y_n, y_{n-1}) = \mathbf{1}(y_n < y_{n-1})$$

$$\widehat{\delta} := \frac{1}{\mathcal{Z}}(|y_n - y^*|\eta_1(y_n) + |y_n - y_{n-1}|\eta_2(y_n, y_{n-1}))$$

$$\eta(y) = \mathbf{1}(\widehat{\delta} > 1/2)$$

Proposition 6. *For the instantaneous model, the observation threshold reads*

$$y^* = b_{i,1}\lambda(r\frac{1-\alpha}{\alpha} + \frac{1}{ln(2)})$$

Proof. The general expression of $x^0 := \frac{1}{b_{i,1}}x_i^t$ when $n \equiv 0 \, mod(r)$ is:

$$x^0 = \sum_{p=1}^{\infty}(1-\alpha)^p\sum_{k=0}^{r-1}u_{rp+k} + u_0$$

where all the u_i are realizations of independent Poisson variables of parameter λ. Similarly:

$$x^1 = \sum_{p=1}^{\infty}(1-\alpha)^p\sum_{k=0}^{r-1}u_{rp+k} + (u_0 + u_1)$$

The first term only depends on what happened before the last opening of the door and has a mean:

$$\mathbb{E}(\sum_{p=1}^{\infty}(1-\alpha)^p\sum_{k=0}^{r-1}u_{rp+k}) = r\lambda\frac{1-\alpha}{\alpha}$$

Then we are in the same configuration than for the propagation model case. □

In the next section, we assume that the mode is perfectly known and focus on the error analysis given sparse available data.

4. ESTIMATION ERROR ANALYSIS

In this section, we present analytical results on how the model propagates error and the influence of possible observations. Let us assume that we have one sensor monitoring cell c.

Error definition.

Let $\tilde{\mathbf{x}}_n$ denote the true state and $\mathbf{x}_n$ denote the state estimate, computed with the same model and including the observations. We consider the mean square error for quantities of interest.

Definition 1. *The local empirical error in cell i is given by:*

$$\widehat{\Delta}_I^2(i,c) := \frac{1}{N}\sum_{t=1}^{I}(x_n^i - \tilde{x}_n^i)^2$$

In order to derive analytical results, we make the following assumptions on the error convergence.

Assumption 1.

$$\widehat{\Delta}_I^2(i,c) \underset{I\to\infty}{\overset{\mathbb{P}}{\to}} (\lim_{n'\to\infty} \mathbb{E}[\frac{1}{r}\sum_{n=n'}^{n'+r-1}(x_n^i - \tilde{x}_n^i)^2])$$

The assumption is justified by the following points.

- the errors are additive thus $\mathbf{x}_n - \tilde{\mathbf{x}}_n$ is independent of the sequence $(\mathbf{u}_n)_n$,
- the error terms $\mathbf{w}$ are independent and identically distributed,
- the sequence $(A_n)_n$ is periodic with period r and $\exists \rho < 1$ such that $|||\prod_{n=1}^{r} A_n||| \leq \rho$,

thus the distribution of the sequence $(\mathbf{x}_n - \tilde{\mathbf{x}}_n)$ tends to a periodic regime with period r. Cesaro's Theorem and the strong law of large numbers yield the result.

In the following section we study error propagation properties for the propagation model and for the instantaneous model, using the following error definitions.

Definition 2. *Local error in cell i:*

$$\Delta^2(i,c) := \underset{n'\to\infty}{lim}\ \mathbb{E}[\frac{1}{r}\sum_{n=n'}^{n'+r-1}(x_n^i - \tilde{x}_n^i)^2]$$

Global error:

$$\Delta^2(c) := \sum_i \Delta^2(i,c)$$

4.1 Propagation model

In the propagation model, pedestrians are moving only from the left to the right and all the errors on the model are independent (Q is diagonal). Thus, the error in cell i only depends of what happens in the left cells $\{1,\cdots,i\}$. If there is no sensor, the error on the propagation accumulates across the entire corridor. If there is a sensor in cell c, the propagation error accumulates from the left-most cell to cell c, and then from cell c to the end of the corridor.

We recall here that all errors are independent and normally distributed with variance σ_u^2 for the entry cell, σ_W^2 for the other cells and σ_R^2 for the sensors. In order to simplify the expressions, we set the indexes of the positions from 0 to $I-1$ (instead of from 1 to I).

Proposition 7. *Under the previous hypothesis, if $0 < c < I$:*

$$\begin{aligned}\Delta^2(i,c) &= (\sigma_u^2 + i\sigma_W^2)\mathbf{1}(i<c)\\ &+ [\frac{(\sigma_u^2 + c\sigma_W^2)\sigma_R^2}{(\sigma_u^2 + c\sigma_W^2)+\sigma_R^2} + \sigma_W^2(i-c)]\mathbf{1}(I > i \geq c)\\ &+ \frac{r+1}{2}[\frac{(\sigma_u^2 + c\sigma_W^2)\sigma_R^2}{(\sigma_u^2 + c\sigma_W^2)+\sigma_R^2} + \sigma_W^2(I-c)]\mathbf{1}(i=I)\end{aligned}$$

if $c = 0$:

$$\Delta^2(i,c) = \frac{\sigma_u^2\sigma_R^2}{\sigma_u^2+\sigma_R^2} + i\sigma_W^2$$

if $c = I$:

$$\Delta^2(i,c) = (\sigma_u^2 + i\sigma_W^2)\mathbf{1}(i<I) + [\frac{(\frac{r+1}{2}(\sigma_u^2+I\sigma_W^2))\sigma_R^2}{(\frac{r+1}{2}(\sigma_u^2+I\sigma_W^2))+\sigma_R^2}]\mathbf{1}(i=I)$$

Proof. Let us first consider the case were there is no sensor:
for $i = 0$,

$$\mathbb{E}[(x_n^0 - \tilde{x}_n^0)^2] = \sigma_u^2$$

for $0 < i < I$

$$\begin{aligned}\mathbb{E}[(x_n^i - \tilde{x}_n^i)^2] &= \mathbb{E}[(x_{n-1}^i - \tilde{x}_{n-1}^i)^2] + \sigma_W^2\\ &= i\sigma_W^2 + \sigma_u^2\end{aligned}$$

for $i = I$

$$\begin{aligned}\mathbb{E}[(x_n^I - \tilde{x}_n^I)^2] &= \frac{1}{r}\sum_{k=1}^{r}\sum_{j=1}^{k}[\mathbb{E}[(x_{n-j}^{I-1} - \tilde{x}_{n-j}^{I-1})^2] + \sigma_W^2]\\ &= \frac{1}{r}\sum_{k=1}^{r} k(\sigma_u^2 + I\sigma_W^2)\\ &= \frac{r+1}{2}(\sigma_u^2 + I\sigma_W^2)\end{aligned}$$

By using equation (1) in $i = c$ when there is a sensor and updating the results for $i > c$ with the same calculation, we obtain the result. □

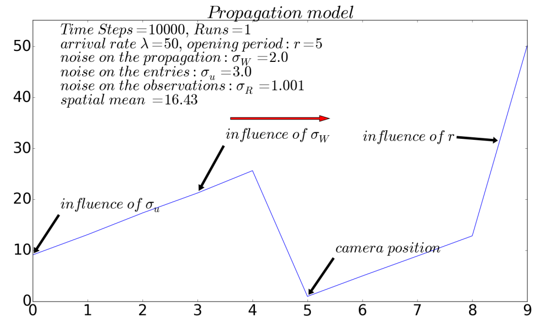

Figure 6: Local estimation error: for the propagation model with sensor placed in cell 5.

Results on the overall error follow immediately.

Proposition 8. *Under the previous hypothesis, if $0 < c < I$:*

$$\begin{aligned}\Delta^2(c) = \sigma_u^2 c + \sigma_W^2\frac{c^2+(I-c)^2}{2} + \sigma_R^2(I-c)\frac{(\sigma_u^2 + c\sigma_W^2)}{(\sigma_u^2 + c\sigma_W^2)+\sigma_R^2}\\ + \frac{r+1}{2}[\frac{(\sigma_u^2 + c\sigma_W^2)\sigma_R^2}{(\sigma_u^2 + c\sigma_W^2)+\sigma_R^2} + \sigma_W^2(I-c)]\end{aligned}$$

if $c = 0$:

$$\Delta^2(0) = \frac{\sigma_u^2\sigma_R^2}{\sigma_u^2+\sigma_R^2} + \frac{I(I+1)}{2}\sigma_W^2$$

if $c = I$:

$$\Delta^2(I) = \sigma_u^2 I + \sigma_W^2\frac{I^2}{2} + \sigma_R^2\frac{(\sigma_W^2 + \frac{r+1}{2}(\sigma_u^2+I\sigma_W^2))}{(\sigma_W^2 + \frac{r+1}{2}(\sigma_u^2+I\sigma_W^2))+\sigma_R^2}$$

4.2 Sensor placement for propagation model

With the previous formula, we can find the optimal position c^* by performing I evaluations:

$$c^* = \underset{c}{argmin}\Delta^2(c)$$

When one parameter is dominant, c^* for the first sensor takes simple expressions:

- if the error on the entries σ_u is dominant: the sensor should be placed on the left: $c^* = 0$
- if the error on the propagation σ_W is dominant: the sensor should be placed in the center: $c^* = I/2$ for $I \gg r$
- $\Delta^2(c) \underset{r\to\infty}{\to} \infty$ except for $c^* = I$
- if $\sigma_u = \sigma_W = 0$, the model is perfect and there is no point in placing a sensor

Remark 3. *As the error is only additive, unless the variances depend on λ, the error function is independent of the entries.*

$\sigma_W = 1.0$	$\sigma_u = 0.0$	$\sigma_u = 1.0$
$\sigma_R = 0.001$	5	5
$\sigma_R = 1.0$	5	6

$\sigma_W = 0.0$	$\sigma_u = 1.0$
$\sigma_R = 0.001$	0
$\sigma_R = 1.0$	0

Figure 7: Optimal positioning: when the model dynamics error dominates the error on the inflow, it is optimal to place the sensor around the middle of the corridor (left). When the inflow error dominates the model dynamics error it is optimal to place the sensor around the entrance.

$\sigma_W = 1.0$	$\sigma_u = 0.0$	$\sigma_u = 1.0$
$\sigma_R = 0.001$	(5,9)	(4,9)
$\sigma_R = 1.0$	(6,9)	(4,9)

$\sigma_W = 0.0$	$\sigma_u = 1.0$
$\sigma_R = 0.001$	(0,0)
$\sigma_R = 1.0$	(0,0)

Figure 8: Optimal positioning: in the case of two sensors.

Example.
We present in the tables above an example illustrating the dependency of the optimal placement to the comparative model error and inflow error. We also produce optimal positions in the case of the placement of two sensors.

4.3 Sensor placement for instantaneous model

For the instantaneous model, we can derive similar formulas in the case where Q is diagonal:

PROPOSITION 9. *If Q is a diagonal matrix:*

$$\Delta^2(i,c) = Q_{ii}\beta\mathbf{1}(i \neq c) + \frac{Q_{ii}\beta\sigma_R^2}{Q_{ii}\beta + \sigma_R^2}\mathbf{1}(i = c) \qquad (10)$$

$$(11)$$

where:

$$\beta = \frac{r+1}{2} + r\frac{1-\alpha}{\alpha}$$

And:

$$\sum_i \Delta^2(i,c) = (\sum_i \beta Q_{ii}) - (\beta Q_{cc}[1 - \frac{\beta\sigma_R^2}{Q_{cc}\beta + \sigma_R^2}])$$

PROOF. If there is no sensor in i:

$$\begin{aligned}\mathbb{E}[(x_i^t - \tilde{x}_i^t)^2] &= \frac{1}{r}\sum_{k=0}^{r-1}[\sum_{j=0}^{k} Q_{ii} + \sum_{p=1}(1-\alpha)^p \sum_{j=0}^{r-1} Q_{ii}] \\ &= Q_{ii}[\frac{r+1}{2} + r\frac{1-\alpha}{\alpha}]\end{aligned}$$

If there is a sensor in i, use equation (1). □

Thus, in this case:

- the sensor only has a local influence
- the error does not accumulate in space but in time
- the optimal position is where the uncertainty is maximal:

$$\begin{aligned} & c^* \in \underset{i}{argmax}(i \mapsto Q_{ii}[1 - \frac{\beta\sigma_R^2}{Q_{ii}\beta + \sigma_R^2}]) \\ \Leftrightarrow\ & c^* \in \underset{i}{argmax}(Q_{ii})\end{aligned}$$

- the error can be split into a difference between the error made when there is no observation and a reduction performed thanks to the observation

5. CONCLUSION

In this work, we introduced a novel generative model for instantaneous crowd levels. We provided analytical results on the case of a corridor with pedestrians moving from left to right and unknown status of the right hand side exit. We have proven that our framework is equivalent to the natural dynamical propagation framework under the proper choice of parameters, but exhibits a number of advantages in terms of estimation error, calibration issues, and stability properties.

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