Errata of "Synchronization in Complex Networks of Nonlinear Dynamical Systems", World Scientific, 2007

Chai Wah Wu

1. In the statement of Theorem 4.8 on page 55, $U(G-\mu I) \preceq 0$ **should be changed to** $U(G-\mu I) \succeq 0$**.**

2. In the statement of Corollary 4.11 on page 56, it should read: *If in addition* $G+G^T \in \mathcal{W}$ *and is irreducible, then* $\mu(G) \geq a_1(G) > 0$.

3. On page 86, Equation (6.6) should read:

$$x(k+1) = (M(k) \otimes D(k))\, x(k) + \mathbf{1} \otimes u(k)$$

4. The proof of Theorem 6.44 starting from the last sentence on page 102 should be corrected as follows:

In this case $x \notin X^*$ and $d(x, X^*) \leq \|x\| = d(x, Z^*)$ since $0 \in X^*$. It is clear that $y = Ax$ can be written as

$$y = \begin{pmatrix} * \\ \vdots \\ * \\ -ra_1e_1 \\ ra_2e_2 \end{pmatrix}$$

Let βe be a projection vector of y onto X^*. By the weak monotonicity of the norm,

$$d(y, X^*) = \|y - \beta\mathbf{1}\| \geq \left\| \begin{pmatrix} 0 \\ \vdots \\ 0 \\ (-ra_1 - \beta)e_1 \\ (ra_2 - \beta)e_2 \end{pmatrix} \right\| = \left\| r\left(x - \frac{\beta}{r}z\right) \right\|$$

Since 0 is a projection vector of x onto Z^*, this implies that $\|x - \frac{\beta}{r}z\| \geq d(x, Z^*)$ and

$$d(y, X^*) \geq |r|d(x, Z^*) \geq |r|d(x, X^*) \geq d(x, X^*)$$

Thus A is not set-contractive. □

5. On page 104, the statement and proof of Theorem 6.46 should be corrected as:

Theorem 1 *Let A be an n by n constant row sum matrix and K be an n by $n-1$ matrix whose columns form a orthonormal basis of $\mathbf{1}^\perp$. Then $c(A) = \left\|\left(A - \frac{\mathbf{1}\mathbf{1}^T}{n}A\right)K\right\|_2 \leq \|AK\|_2$ with respect to $\|\cdot\|_2$ and $X^* = \{\alpha\mathbf{1} : \alpha \in \mathbb{R}\}$. In particular $\left\|\left(A - \frac{\mathbf{1}\mathbf{1}^T}{n}A\right)K\right\|_2 \leq 1$ if and only if A is set-nonexpanding with respect to $\|\cdot\|_2$ and $X^* = \{\alpha\mathbf{1} : \alpha \in \mathbb{R}\}$. Similarly, $\left\|\left(A - \frac{\mathbf{1}\mathbf{1}^T}{n}A\right)K\right\|_2 < 1$ if and only if A is set-contracting with respect to $\|\cdot\|_2$ and $X^* = \{\alpha\mathbf{1} : \alpha \in \mathbb{R}\}$.*

Proof: Define $J = \frac{1}{n}\mathbf{1}\mathbf{1}^T$ as the n by n matrix where each entry is of value $\frac{1}{n}$. Note that $\|x\|_2 = \|Kx\|_2$ and $JK = 0$. Let $B = A - JA$. Then

$$\|BK\|_2 = \max_{\|x\|_2=1} \|BKx\|_2 = \max_{\|Kx\|_2=1} \|BKx\|_2 = \max_{x\perp\mathbf{1}, \|x\|_2=1} \|Bx\|_2$$

By Lemma 6.33 $P(x) = Jx$ and $d(Ax, X^*) = \|Bx\|_2$. Since A has constant row sums, $A(X^*) \subseteq X^*$ and by Lemma 6.40 $c(A) = \max_{P(x)=0,\|x\|_2=1} d(Ax, X^*) = \max_{P(x)=0,\|x\|_2=1} \|Bx\|_2$. Since $P(x) = 0$ if and only if $x\perp\mathbf{1}$, this means that $c(A) = \|BK\|_2$. Note that $d(Ax, X^*) \leq d(Ax, P(x)) = \|(A-J)x\|_2$, and coupling this with the fact that $\|AK\|_2 = \|(A-J)K\|_2 = \max_{P(x)=0,\|x\|_2=1} \|(A-J)x\|_2$ we can show that $c(A) \leq \|AK\|_2$. □

6. The statement and proof of Theorem 6.48 should be corrected as:

Theorem 2 *Let A be an n by n constant row sum matrix and K be as defined in Theorem 1. Let w be a positive vector such that $\max_i w_i \leq 1$ and $W = diag(w)$. Then $c(A) \leq \left\|W^{\frac{1}{2}}\left(A - \frac{\mathbf{1}w^T}{\sum_i w_i}A\right)W^{-1}K\right\|_2$ with respect to $\|\cdot\|_w$ and $X^* = \{\alpha\mathbf{1} : \alpha \in \mathbb{R}\}$.*

Proof: The proof is similar to Theorem 6.46. Define $J_w = \frac{\mathbf{1}w^T}{\sum_i w_i}$ and $B = A - J_wA$. Note that $J_wW^{-1}K = 0$. Then

$$\|W^{\frac{1}{2}}BW^{-1}K\|_2 = \max_{\|Kx\|_2=1} \|W^{\frac{1}{2}}BW^{-1}Kx\|_2 = \max_{x\perp\mathbf{1},\|x\|_2=1} \|W^{\frac{1}{2}}BW^{-1}x\|_2$$

Now $x \perp \mathbf{1}$ if and only if $W^{-1}x \perp w$. Since $\|x\|_2 = \|W^{-\frac{1}{2}}x\|_w$, this means that $\|W^{\frac{1}{2}}BW^{-1}K\|_2 = \max_{x\perp w, \|W^{\frac{1}{2}}x\|_w=1} \|W^{\frac{1}{2}}Bx\|_2$. Since $\max_i w_i \leq 1$, this means that $\|W^{\frac{1}{2}}x\|_w = \sqrt{\sum_i (w_i x_i)^2} \leq \|x\|_w$ and thus

$$\|W^{\frac{1}{2}}BW^{-1}K\|_2 \geq \max_{x\perp w, \|x\|_w=1} \|W^{\frac{1}{2}}Bx\|_2$$

It is straightforward to show that $P(x) = J_w x$ and thus $d(Ax, X^*) = \|Bx\|_w = \|W^{\frac{1}{2}}Bx\|_2$. Since A has constant row sums, $A(X^*) \subseteq X^*$ and by Lemma 6.40 $c(A) = \max_{P(x)=0, \|x\|_w=1} d(Ax, X^*) = \max_{P(x)=0, \|x\|_w=1} \left\|W^{\frac{1}{2}}Bx\right\|_2$. Since $P(x) = 0$ if and only if $x \perp w$, this means that $c(A) \leq \left\|W^{\frac{1}{2}}BW^{-1}K\right\|_2$.
□

7. Page 105, 4 lines from bottom. $\|A_2K\|_2 = 1$ should be changed to $\|(A_2 - JA_2)\,K\|_2 = 1$.

8. Page 106, 4 lines from top. $\|A_3K\|_2 = 0.939$ should be changed to $\|(A_3 - JA_3)\,K\|_2 = 0.866$.

9. The example A_4 on page 106 should be corrected to:

The stochastic matrix

$$A_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0.9 & 0.1 & 0 \\ 0.01 & 0.01 & 0.98 \end{pmatrix}$$

has an interaction digraph that contains a spanning directed tree. However, it is not set-nonexpanding with respect to $\|\cdot\|_2$ and $X^* = \{\alpha\mathbf{1} : \alpha \in \mathbb{R}\}$ since $\|(A_4 - JA_4)\,K\|_2 = 1.1102 > 1$. This shows that the converse of Theorem 6.44 is not true for $\|\cdot\|_2$. The matrix A_3 shows that the converse of Theorem 6.44 is also false for stochastic matrices with respect to $\|\cdot\|_\infty$ and X^*. On the other hand, A_4 is set-contractive with respect to $\|\cdot\|_\infty$ and X^* since A_4 is a scrambling matrix. Furthermore, A_4 is set-contractive with respect to $\|\cdot\|_w$ and X^* for $w = (1, 0.0527, 1)^T$ since $\|W^{\frac{1}{2}}\,(A_4 - J_w A_3)\,W^{-1}K\|_2 = 0.9974 < 1$.