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PROBABILITIES ON FINITE MODELS[1]

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§1. Introduction. Let $\mathscr{S}$ be a finite set of (nonlogical) predicate symbols. By an $\mathscr{S}$-structure, we mean a relational structure appropriate for $\mathscr{S}$. Let $\mathscr{A}_n(\mathscr{S})$ be the set of all $\mathscr{S}$-structures with universe $\{1, \cdots, n\}$. For each first-order $\mathscr{S}$-sentence σ (with equality), let $\mu_n(\sigma)$ be the fraction of members of $\mathscr{A}_n(\mathscr{S})$ for which σ is true. We show that $\mu_n(\sigma)$ always converges to 0 or 1 as $n \to \infty$, and that the rate of convergence is geometrically fast. In fact, if T is a certain complete, consistent set of first-order $\mathscr{S}$-sentences introduced by H. Gaifman [6], then we show that, for each first-order $\mathscr{S}$-sentence σ, $\mu_n(\sigma) \to_n 1$ iff $T \models \sigma$. A surprising corollary is that each finite subset of T has a finite model. Following H. Scholz [8], we define the *spectrum* of a sentence σ to be the set of cardinalities of finite models of σ. Another corollary is that for each first-order $\mathscr{S}$-sentence σ, either σ or $\sim\sigma$ has a cofinite spectrum (in fact, either σ or $\sim\sigma$ is "nearly always" true).

Let $\mathscr{B}_n(\mathscr{S})$ be a subset of $\mathscr{A}_n(\mathscr{S})$ which contains for each $\mathfrak{A}$ in $\mathscr{A}_n(\mathscr{S})$ exactly one structure isomorphic to $\mathfrak{A}$. For each first-order $\mathscr{S}$-sentence σ, let $\nu_n(\sigma)$ be the fraction of members of $\mathscr{B}_n(\mathscr{S})$ for which σ is true. By making use of an asymptotic estimate [3] of the cardinality of $\mathscr{B}_n(\mathscr{S})$ and by our previously mentioned results, we show that $\nu_n(\sigma)$ converges as $n \to \infty$, and that $\lim_n \nu_n(\sigma) = \lim_n \mu_n(\sigma)$. If $\mathscr{S}$ contains at least one predicate symbol which is not unary, then the rate of convergence is geometrically fast.

The author thanks R. W. Robinson for bringing Oberschelp's preliminary work [7] on the cardinality of $\mathscr{B}_n(\mathscr{S})$ to his attention, and Rolando Chuaqui for pointing out to him that Carnap [1] introduced the idea of considering $\lim_n \nu_n(\sigma)$. Carnap proved, in the special case when $\mathscr{S}$ contains only unary predicate symbols, that $\lim_n \nu_n(\sigma)$ exists, and is 0 or 1.

§2. The probabilities μ_n. Let $\mathscr{S}$ be a finite set of predicate symbols, let $\mathscr{A}_n(\mathscr{S})$ be as before, and let $a_n(\mathscr{S})$ be the cardinality of $\mathscr{A}_n(\mathscr{S})$. For example, if $\mathscr{S} = \{P\}$, where P is a binary predicate symbol, then $a_n(\mathscr{S}) = 2^{n^2}$.

Let $\mathbf{1}, \mathbf{2}, \mathbf{3}, \cdots$ be distinct constant symbols, and let $\mathscr{S}' = \mathscr{S} \cup \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \cdots\}$. For each $\mathfrak{A}$ in $\mathscr{A}_n(\mathscr{S})$, let $\mathfrak{A}'$ be the inessential expansion of $\mathfrak{A}$ to $\mathscr{S}'$ in which the interpretation of $\boldsymbol{i}$ is i, if $i \leq n$, and n otherwise.

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Let $\mu_n^{\mathscr{S}}$ be the probability measure on $\mathscr{A}_n(\mathscr{S})$ which weights each structure equally. Thus, if $\mathscr{S} = \{P\}$, with P a binary predicate symbol, then

$$\mu_n^{\mathscr{S}}(\{\mathfrak{A} \in \mathscr{A}_n(\mathscr{S}) : \mathfrak{A}' \models P12\}) = 1/2 \quad \text{if } n \geq 2,$$
$$\mu_n^{\mathscr{S}}(\{\mathfrak{A} \in \mathscr{A}_n(\mathscr{S}) : \mathfrak{A}' \models P21 \wedge P23 \wedge P31\}) = 1/8 \quad \text{if } n \geq 3.$$

If σ is a first-order $\mathscr{S}'$-sentence, then abbreviate $\mu_n^{\mathscr{S}}(\{\mathfrak{A} \in \mathscr{A}_n(\mathscr{S}) : \mathfrak{A}' \models \sigma\})$, the fraction of members $\mathfrak{A}$ of $\mathscr{A}_n(\mathscr{S})$ for which $\mathfrak{A}'$ is a model of σ, by $\mu_n(\sigma)$. This is unambiguous, since if both $\mathscr{S}$ and $\mathscr{T}$ contain the predicate symbols that appear in σ, then it is not hard to see that

$$\mu_n^{\mathscr{S}}(\{\mathfrak{A} \in \mathscr{A}_n(\mathscr{S}) : \mathfrak{A}' \models \sigma\}) = \mu_n^{\mathscr{T}}(\{\mathfrak{A} \in \mathscr{A}_n(\mathscr{S}) : \mathfrak{A}' \models \sigma\}).$$

Note that $\mu_n(\sigma)$ is defined for first-order $\mathscr{S}'$-sentences σ, whereas $\mu_n^{\mathscr{S}}(\mathscr{A})$ is defined only for (certain) sets $\mathscr{A}$ of $\mathscr{S}$-structures.

Let $X = \{x_1, \cdots, x_m\}$ be a finite set of m distinct individual variables. Then a *complete open description* $M(x_1, \cdots, x_m)$ is a conjunction $\bigwedge\{\phi : \phi \in A\}$, where for each k-ary P in $\mathscr{S}$ and each k-tuple $\langle z_1, \cdots, z_k\rangle$ of members of X, the set A contains exactly one of $Pz_1 \cdots z_k$ or $\sim Pz_1 \cdots z_k$. Note that the equality symbol $=$ does not appear in M. We abbreviate $\langle x_1, \cdots, x_m\rangle$ by $\boldsymbol{x}$ (we also abbreviate the formula $\forall x_1 \cdots \forall x_m \phi$ by $\forall \boldsymbol{x}\phi$.) The complete open description $N(\boldsymbol{x}, y)$ is said to *extend* the complete open description $M(\boldsymbol{x})$ if y is a new variable distinct from all the x_i, and if every conjunct of $M(\boldsymbol{x})$ is a conjunct of $N(\boldsymbol{x}, y)$.

Let T be the set of all $\mathscr{S}$-sentences

$$(1) \qquad \forall \boldsymbol{x}\Big(\Big(\Big(\bigwedge_{i \neq j} x_i \neq x_j\Big) \wedge M(\boldsymbol{x})\Big) \rightarrow \exists y\Big(\Big(\bigwedge_i y \neq x_i\Big) \wedge N(\boldsymbol{x}, y)\Big)\Big),$$

where $M(\boldsymbol{x})$ is a complete open description, and where $N(\boldsymbol{x}, y)$ is a complete open description extending $M(\boldsymbol{x})$. A set of sentences equivalent to T was introduced by H. Gaifman [6]. In [6], Gaifman shows the following:

THEOREM 1 (GAIFMAN). *T is consistent and complete.*

PROOF. One way to show that T is consistent is to create an infinite model of T in ω steps. We will show later (Corollary 5) that surprisingly enough, every finite subset of T has a finite model. By the compactness theorem of first-order logic, this is certainly enough to guarantee that T is consistent. We will now sketch Gaifman's proof that T is complete. The Łoś-Vaught test [9] says that if T has no finite models, and if every two countable models of T are isomorphic, then T is complete. Clearly, T has no finite models. To show that every two countable models $\mathfrak{A}$ and $\mathfrak{B}$ of T are isomorphic, we use Cantor's "back-and-forth" argument, in which we extend an isomorphism between finite substructures $\mathfrak{A}_1 \subseteq \mathfrak{A}$ and $\mathfrak{B}_1 \subseteq \mathfrak{B}$ by adding one more element to one, and then finding a suitable element in the other by which the isomorphism can be extended. In fact, T is defined in precisely such a way as to make this "back-and-forth" argument work. □

Gaifman introduces T in the context of a certain infinite "probability model" (which was suggested to him by Rabin and Scott), that seems on the surface to be intimately related to what we are doing. However, Gaifman never considers finite structures, and our results do not seem to follow from his.

If we let $\mu(\sigma) = \lim_n \mu_n(\sigma)$, which we will show to exist, then μ is a probability measure on sentences, in the sense of Gaifman, and coincides with the probability measure in Rabin and Scott's example. For details, see [6].

The next theorem is the key tool in this section.

THEOREM 2. *If* $\sigma \in T$, *then* $\mu_n(\sigma) \to_n 1$.

PROOF. Assume for convenience that $\mathscr{S} = \{P\}$, where P is a binary predicate symbol. The modification of the proof in the general case is clear. Let σ be the sentence (1), and let $\phi(x)$ be

$$M(x) \wedge \forall y\Big(\Big(\bigwedge_i y \neq x_i\Big) \to \sim N(x, y)\Big).$$

Let $\psi(y)$ be $\sim(A_1 \wedge \cdots \wedge A_{2m+1})$, where for each i $(1 \leq i \leq m)$,

$$A_{2i-1} = \begin{cases} Piy & \text{if } Px_i y \text{ is a conjunct of } N(x, y),\\ \sim Piy & \text{if } \sim Px_i y \text{ is a conjunct of } N(x, y),\end{cases}$$

$$A_{2i} = \begin{cases} Pyi & \text{if } Pyx_i \text{ is a conjunct of } N(x, y),\\ \sim Pyi & \text{if } \sim Pyx_i \text{ is a conjunct of } N(x, y),\end{cases}$$

$$A_{2m+1} = \begin{cases} Pyy & \text{if } Pyy \text{ is a conjunct of } N(x, y),\\ \sim Pyy & \text{if } \sim Pyy \text{ is a conjunct of } N(x, y).\end{cases}$$

Assume that $n > m$. Then

$$\begin{aligned}
\mu_n(\sim\sigma) &= \mu_n\Big(\exists x_1 \cdots \exists x_m\Big(\bigwedge_{i \neq j} x_i \neq x_j \wedge M(x) \wedge \forall y\Big(\Big(\bigwedge_i y \neq x_i\Big) \to \sim N(x, y)\Big)\Big)\Big)\\
&= \mu_n\Big(\exists x_1 \cdots \exists x_m\Big(\bigwedge_{i \neq j} x_i \neq x_j \wedge \phi(x)\Big)\Big)\\
&\leq \sum \{\mu_n(\phi(\mathbf{a}_1, \cdots, \mathbf{a}_m)) : 1 \leq a_i \leq n, \text{ and } a_i \neq a_j \text{ if } i \neq j, \text{ for }\\
&\qquad 1 \leq i \leq m,\ 1 \leq j \leq m\}, \text{ where we substitute } \mathbf{a}_i \text{ for } x_i\\
&= \sum \mu_n(\phi(\mathbf{1}, \cdots, \mathbf{m})) \quad \text{by symmetry}\\
&= n(n-1)\cdots(n-m+1)\mu_n(\phi(\mathbf{1}, \cdots, \mathbf{m})) \leq n^m \mu_n(\phi(\mathbf{1}, \cdots, \mathbf{m}))\\
&\leq n^m \mu_n\Big(\forall y\Big(\Big(\bigwedge_i y \neq \mathbf{i}\Big) \to \psi(y)\Big)\Big),\\
&\qquad \text{since } \models\Big(\phi(\mathbf{1}, \cdots, \mathbf{m}) \to \forall y\Big(\Big(\bigwedge_i y \neq \mathbf{i}\Big) \to \psi(y)\Big)\Big)\\
&= n^m \prod_{j=m+1}^{n} \mu_n \psi(\mathbf{j}) \quad \text{by independence}\\
&= n^m(1 - (1/2^{2m+1}))^{n-m} \quad \text{by an obvious computation}\\
&= n^m k^{n-m}, \quad \text{where } k = (1 - (1/2^{2m+1})) < 1\\
&\to 0 \text{ as } n \to \infty, \quad \text{by l'Hospital's rule.}
\end{aligned}$$

Since $\mu_n(\sigma) + \mu_n(\sim\sigma) = 1$, we have $\mu_n(\sigma) \to_n 1$. □

COROLLARY 3. *If* σ *is a first-order* $\mathscr{S}$*-sentence, then* $T \models \sigma$ *iff* $\mu_n(\sigma) \to_n 1$.

PROOF. $\Rightarrow$: Assume that $T \models \sigma$. By the compactness theorem, there is a finite subset $\{\sigma_1, \cdots, \sigma_r\} \subseteq T$ such that $\models((\sigma_1 \wedge \cdots \wedge \sigma_r) \to \sigma)$; and hence $\models(\sim\sigma \to (\sim\sigma_1 \vee \cdots \vee \sim\sigma_r))$. Therefore, $\mu_n(\sim\sigma) \leq \mu_n(\sim\sigma_1 \vee \cdots \vee \sim\sigma_r) \leq \mu_n(\sim\sigma_1) + \cdots + \mu_n(\sim\sigma_r) \to_n 0$. Since $\mu_n(\sigma) + \mu_n(\sim\sigma) = 1$, it follows that $\mu_n(\sigma) \to_n 1$.

$\Leftarrow$: Assume that $\mu_n(\sigma) \to_n 1$ and $T \nvDash \sigma$. Since T is complete, $T \vDash \sim\sigma$. But as we just saw, if $T \vDash \sim\sigma$, then $\mu_n(\sim\sigma) \to_n 1$. Therefore $\mu_n(\sigma) \to_n 0$, a contradiction. □

We can now prove the main result of this section, which does not involve (in its statement) the set T.

THEOREM 4. *If σ is a first-order sentence with no function or constant symbols, then $\mu_n(\sigma)$ converges to either 0 or 1.*

PROOF. Either $T \vDash \sigma$ or $T \vDash \sim\sigma$. If $T \vDash \sigma$, then $\mu_n(\sigma) \to_n 1$. If $T \vDash \sim\sigma$, then $\mu_n(\sim\sigma) \to_n 1$, so $\mu_n(\sigma) \to_n 0$. □

We now prove two further corollaries of Theorem 2.

COROLLARY 5. *If T' is any finite subset of T, then T' has a finite model. In fact, $(\exists N)(\forall n > N)(T'$ has a model of cardinality $n)$.*

PROOF. If T' has r sentences, then find N so large that $\mu_n(\sim\sigma) < 1/r$ for each $n > N$ and each σ in T'. Then $\mu_n(\sim\sigma_1 \vee \cdots \vee \sim\sigma_r) \leq \mu_n(\sim\sigma_1) + \cdots + \mu_n(\sim\sigma_r) < 1$, and so $\mu_n(\sigma_1 \wedge \cdots \wedge \sigma_r) > 0$. □

COROLLARY 6. *If σ is a first-order sentence, then either σ or $\sim\sigma$ has a cofinite spectrum.*

PROOF 1. For each first-order sentence σ, there is a sentence σ^* which has no function or constant symbols, but which has the same spectrum as σ (the sentence σ^* is derived from σ by well-known techniques of replacing functions by relations). Therefore, by replacing σ with σ^*, we can assume that σ has only predicate symbols. So, either $\mu_n(\sigma) \to_n 1$ or $\mu_n(\sim\sigma) \to_n 1$. If $\mu_n(\sigma) \to_n 1$, then in particular $\exists N(\forall n > N)(\mu_n(\sigma) > 0)$; hence σ has a model of cardinality n for each $n > N$. Similarly, if $\mu_n(\sim\sigma) \to_n 1$. □

PROOF 2. Assume that σ is a $\mathscr{T}$-sentence, where $\mathscr{T}$ is a finite set of predicate symbols, function symbols, and constant symbols. Let Σ contain the following sentences:

$\exists x_1 \cdots \exists x_k(\bigwedge_{i \neq j} x_i \neq x_j)$, for each positive integer k;
$\forall x(F(x) = x_1)$ for each function symbol F in $\mathscr{T}$;
$\forall x Rx$ for each predicate symbol R in $\mathscr{T}$;
$c_i = c_j$ for each pair of constant symbols c_i, c_j in $\mathscr{T}$.

Then Σ is complete by the Łoś-Vaught test [9]. So, either $\Sigma \vDash \sigma$ or $\Sigma \vDash \sim\sigma$. Say $\Sigma \vDash \sigma$. By the compactness theorem, there is a finite subset Σ' of Σ such that $\Sigma' \vDash \sigma$. Let N be larger than any k for which some sentence in Σ' is $\exists x_1 \cdots \exists x_k(\bigwedge_{i \neq j} x_i \neq x_j)$. Then Σ' has a model of cardinality n for each $n > N$. Therefore, σ has a model of cardinality n for each $n > N$. □

Actually, a stronger statement than Corollary 6 is true (if σ contains no function or constant symbols). Namely, either $\exists N(\forall n > N)$ (σ has "many" nonisomorphic models of cardinality n), or $\exists N(\forall n > N)$ ($\sim\sigma$ has "many" nonisomorphic models of cardinality n). We will deal with the trivial case when $\mathscr{S}$ has only unary predicate symbols later. So assume that $\mathscr{S}$ contains at least one predicate symbol which is not unary. Assume that $\mathscr{S}$ contains k_i distinct i-ary predicate symbols. Recall that $a_n(\mathscr{S})$ is the cardinality of $\mathscr{A}_n(\mathscr{S})$; then $a_n(\mathscr{S}) = 2^{\Sigma k_i n^i}$. Assume that σ is a sentence such that $\mu_n(\sigma) \to_n 1$. Then given any θ with $0 \leq \theta < 1$, we can find N such that $\mathscr{A}_n(\mathscr{S})$ contains at least $\theta(2^{\Sigma k_i n^i})$ models of σ for each $n > N$. How many isomorphism types does this represent? Any member of $\mathscr{A}_n(\mathscr{S})$ can be isomorphic to at most $n!$ distinct members of $\mathscr{A}_n(\mathscr{S})$, since there are only $n!$ bijections of $\{1, \cdots, n\}$ onto itself. By Stirling's formula, $n!$ is asymptotic to $(2\pi n)^{1/2}(n/e)^n$. So given θ, with $0 \leq \theta < 1$, there is N such that, for each $n > N$, the number of different non-isomorphic models of σ of cardinality n is at least $\theta(2^{\Sigma k_i n^i})/((2\pi n)^{1/2}(n/e)^n)$. The denominator is dwarfed by 2^{n^i}, if $i \geq 2$, because $n^{(n+1/2)} = 2^{(\log_2 n)(n+1/2)}$. And, some such term 2^{n^i} appears in the numerator. In the next section, we will show that in fact, either σ or $\sim\sigma$ is "nearly always" true when we consider isomorphism types.

§3. **The probabilities ν_n.** As before, let $\mathscr{B}_n(\mathscr{S}) \subseteq \mathscr{A}_n(\mathscr{S})$ contain exactly one representative of each ismorphism type of n-element $\mathscr{S}$-structures. Let $b_n(\mathscr{S})$ be the cardinality of $\mathscr{B}_n(\mathscr{S})$.

Let $\nu_n^{\mathscr{S}}$ be the probability measure on $\mathscr{B}_n(\mathscr{S})$ which weights each structure equally. If σ is a first-order $\mathscr{S}$-sentence, then abbreviate $\nu_n^{\mathscr{S}}(\{\mathfrak{A} \in \mathscr{B}_n(\mathscr{S}) : \mathfrak{A} \models \sigma\})$ by $\nu_n^{\mathscr{S}}(\sigma)$. Here it is not necessarily true that if both $\mathscr{S}$ and $\mathscr{T}$ contain the predicate symbols of σ, then $\nu_n^{\mathscr{S}}(\sigma) = \nu_n^{\mathscr{T}}(\sigma)$; it is easy to find counterexamples even for $n = 2$. However, we will often write ν_n for $\nu_n^{\mathscr{S}}$.

We will show in this section that $\nu_n(\sigma)$ converges as $n \to \infty$, and that $\lim_n \nu_n(\sigma) = \lim_n \mu_n(\sigma)$. The proof is divided into two cases: first, if $\mathscr{S}$ is *monadic* (that is, if $\mathscr{S}$ contains only unary predicate symbols); and second, if $\mathscr{S}$ is not monadic. Carnap [1, p. 567] showed that if $\mathscr{S}$ is monadic, then $\lim_n \nu_n(\sigma)$ exists, and is 0 or 1. For the monadic case, we will make use of the following simple lemma, which is proved in Feller [4].

LEMMA 7. *The number of s-tuples $(x_1, \cdots, x_s)$ of integers x_i such that $x_1 + \cdots + x_s = c$ and $0 \leq x_i \leq c$ for each i is the binomial coefficient $C_{c+s-1,s-1}$.*

PROOF. [4, p. 36]. □

Our main tool in the interesting case, in which $\mathscr{S}$ is not monadic, is the following result. Recall that $a_n(\mathscr{S})$ is the cardinality of $\mathscr{A}_n(\mathscr{S})$, i.e., in graph-theoretic terms, $a_n(\mathscr{S})$ is the number of "labeled" $\mathscr{S}$-structures with universe of cardinality n; and $b_n(\mathscr{S})$ is the cardinality of $\mathscr{B}_n(\mathscr{S})$, i.e., $b_n(\mathscr{S})$ is the number of "unlabeled" $\mathscr{S}$-structures with universe of cardinality n.

THEOREM 8 [3]. *Assume that $\mathscr{S}$ is not monadic. Then $b_n(\mathscr{S})$ is asymptotic to $a_n(\mathscr{S})/n!$, that is,*

$$b_n(\mathscr{S})n!/a_n(\mathscr{S}) \to 1 \quad \text{as } n \to \infty. \tag{2}$$

F. Harary [5] stated (2) without proof for the case when $\mathscr{S}$ contains only one symbol, a binary predicate symbol. W. Oberschelp [7] proved (2) for the case when $\mathscr{S}$ is a set of r-ary predicate symbols for fixed $r \geq 2$. Theorem 8 is a best-possible result, since it is not hard to check that (2) fails if $\mathscr{S}$ is monadic. Theorem 8 was recently proven, independently, by A. Ehrenfeucht (unpublished) and the author [3].

THEOREM 9. *If σ is a first-order sentence with no function or constant symbols, then $\nu_n(\sigma)$ converges as $n \to \infty$, and $\lim_n \nu_n(\sigma) = \lim_n \mu_n(\sigma)$.*

PROOF. *Case* 1. $\mathscr{S}$ is monadic.

Assume that $\mathscr{S} = \{P_1, \cdots, P_m\}$, where each P_i is unary. It is well known that σ is equivalent to a first-order sentence $\bigvee_i \bigwedge_j \tau_{ij}$, where each τ_{ij} is one of two types of sentences:

(A) "There are exactly k points x such that $\bigwedge_{i=1}^m \phi_i(x)$," where each $\phi_i(x)$ is either $P_i x$ or $\sim P_i x$.

(B) The negation of a sentence of type (A).

How many members of $\mathscr{B}_n(\mathscr{S})$ are models of the given sentence of type (A)? For each $\mathfrak{A}$ in $\mathscr{B}_n(\mathscr{S})$, the universe $\{1, \cdots, n\}$ of $\mathfrak{A}$ is partitioned into 2^m sets, where i is in a given set depending on whether $P_t^{\mathfrak{A}}i$ or $\sim P_t^{\mathfrak{A}}i$ for each P_t in $\mathscr{S}$; in fact, the partitioning uniquely characterizes an $\mathscr{S}$-structure up to isomorphism. Assume that $n > k$. From Lemma 7 with $c = n - k$ and $s = 2^m - 1$, the number of members of $\mathscr{B}_n(\mathscr{S})$ which are models of the given sentence of type (A) is $C_{n-k+2^m-2,2^m-2}$, a polynomial in n of degree $2^m - 2$. The number of structures in $\mathscr{B}_n(\mathscr{S})$ altogether is $C_{n+2^m-1,2^m-1}$, a polynomial in n of degree $2^m - 1$, again by Lemma 7.

Therefore, if τ_{ij} is a formula of type (A), then $\nu_n(\tau_{ij}) \to_n 0$. Hence, if τ_{ij} is a formula of type (B), then $\nu_n(\tau_{ij}) \to_n 1$. Therefore,

$$\nu_n\Big(\bigwedge_j \tau_{ij}\Big) \to_n \begin{cases} 0 & \text{if } \tau_{ij} \text{ is of type (A) for some } j, \\ 1 & \text{if } \tau_{ij} \text{ is of type (B) for each } j. \end{cases}$$

And,

$$\nu_n\Big(\bigvee_i \bigwedge_j \tau_{ij}\Big) \to_n \begin{cases} 0 & \text{if } \nu_n\Big(\bigwedge_j \tau_{ij}\Big) \to_n 0 \text{ for each } i, \\ 1 & \text{otherwise.} \end{cases}$$

Therefore, $\nu_n(\bigvee_i \bigwedge_j \tau_{ij})$ converges to 0 or 1, and it converges to 1 iff there is some i such that, for each j, the sentence τ_{ij} is of type (B). But in this case, $T \models \bigvee_i \bigwedge_j \tau_{ij}$, where T is the set of sentences of §2. Then $\mu_n(\bigvee_i \bigwedge_j \tau_{ij}) \to_n 1$.

So, we have seen that if $\mathscr{S}$ is monadic and σ is a first-order $\mathscr{S}$-sentence, then either $\nu_n(\sigma) \to_n 0$ or $\nu_n(\sigma) \to_n 1$, and that $\nu_n(\sigma) \to_n 1$ implies that $\mu_n(\sigma) \to_n 1$. If $\nu_n(\sigma) \to_n 0$, then $\nu_n(\sim\sigma) \to_n 1$, so $\mu_n(\sim\sigma) \to_n 1$, and so $\mu_n(\sigma) \to_n 0$.

Case 2. $\mathscr{S}$ is not monadic.

Assume that $\mu_n(\sigma) \to_n 1$; we will show that $\nu_n(\sigma) \to_n 1$. Let x be the number of members of $\mathscr{A}_n(\mathscr{S})$ which are models of σ, and let y be the number of members of $\mathscr{B}_n(\mathscr{S})$ which are models of σ. Then $y \geq x/(n!)$, as in the argument at the end of the previous section. But $y = \nu_n(\sigma)b_n(\mathscr{S})$, and $x = \mu_n(\sigma)a_n(\mathscr{S})$. Hence $\nu_n(\sigma)b_n(\mathscr{S}) \geq \mu_n(\sigma)a_n(\mathscr{S})/n!$, and so

$$\nu_n(\sigma) \geq \mu_n(\sigma)/(b_n(\mathscr{S})n!/a_n(\mathscr{S})). \tag{3}$$

But the numerator of the right-hand side of (3) converges to 1 by assumption, and the denominator converges to 1 by Theorem 8. Hence $\nu_n(\sigma) \to_n 1$, as desired. So if $\mu_n(\sigma) \to_n 1$, then $\nu_n(\sigma) \to_n 1$. If $\mu_n(\sigma) \to_n 0$, then $\mu_n(\sim\sigma) \to_n 1$, so by the above $\nu_n(\sim\sigma) \to_n 1$, and hence $\nu_n(\sigma) \to_n 0$. □

§4. Additional remarks and counterexamples.

1. If σ is a second-order $\mathscr{S}$-sentence, then $\mu_n^{\mathscr{S}}(\sigma)$ and $\nu_n^{\mathscr{S}}(\sigma)$ need not converge. Let $\mathscr{S} = \varnothing$, let P be a binary predicate symbol, and let σ be

$$\exists P(\forall x \exists ! y(x \neq y \wedge Pxy \wedge Pyx)),$$

where $\exists ! y$ is read "there is exactly one y." Then

$$\mu_n(\sigma) = \nu_n(\sigma) = \begin{cases} 0, & n \text{ odd}, \\ 1, & n \text{ even}. \end{cases}$$

2. What happens if we allow $\mathscr{S}$ to contain function and constant symbols? We can still define $\mathscr{A}_n(\mathscr{S})$ to be the set of all $\mathscr{S}$-structures with universe $\{1, \cdots, n\}$.

Then a constant symbol c can denote any one of n possible values, a 1-place function symbol F can denote any of n^n possible values, and so on. And, we can still define μ_n to be the probability measure on $\mathscr{A}_n(\mathscr{S})$ which weights each structure equally. But if σ is a first-order $\mathscr{S}$-sentence, then $\mu_n(\sigma)$ need no longer converge to 0 or 1. For, if U is a unary predicate symbol, and if c is a constant symbol, then $\mu_n(Uc) = \frac{1}{2}$, and so $\mu_n(Uc) \rightarrow_n \frac{1}{2}$. A more interesting example is given by the sentence $\sigma = \forall x(F(x) \neq x)$. Then

$$\mu_n(\sigma) = (n-1)^n/n^n = (1 - 1/n)^n \rightarrow_n 1/e.$$

It is an open question as to whether, in this extended case, $\mu_n(\sigma)$ and $\nu_n(\sigma)$ necessarily converge, and what the possible limits are.

3. We will now consider the rate of convergence of $\mu_n(\sigma)$ and of $\nu_n(\sigma)$. If $\langle x_n : n \in Z^+ \rangle$ is a sequence, and if $x_n \rightarrow_n x$, then we say that the convergence is *geometrically fast* if there is some θ, with $0 \leq \theta < 1$, and some constant N, such that $|x - x_n| < \theta^n$ for each $n > N$.

Assume that $\mathscr{S}$ contains only predicate symbols. We will show that $\mu_n^{\mathscr{S}}(\sigma)$ converges geometrically fast for each first-order $\mathscr{S}$-sentence σ. We will also show that if $\mathscr{S}$ is not monadic, then $\nu_n^{\mathscr{S}}(\sigma)$ converges geometrically fast for each first-order $\mathscr{S}$-sentence σ. In part 4 of this section, we will give a counterexample to show that this fails if $\mathscr{S}$ is monadic.

THEOREM 10. *If $\mathscr{S}$ contains no function or constant symbols, and if σ is a first-order $\mathscr{S}$-sentence, then $\mu_n^{\mathscr{S}}(\sigma)$ converges geometrically fast. If $\mathscr{S}$ is not monadic, then $\nu_n^{\mathscr{S}}(\sigma)$ converges geometrically fast.*

PROOF. The proof of Theorem 1 shows that if $\sigma \in T$, then there is some k, with $0 \leq k < 1$, and there is some positive integer m, such that $\mu_n(\sim\sigma) < n^m k^{n-m}$, for sufficiently large n. Find $\varepsilon > 0$ small enough that if $\theta = (1 + \varepsilon)k$, then $\theta < 1$. By l'Hospital's rule, $n^{-m}k^m(1 + \varepsilon)^n \rightarrow \infty$ as $n \rightarrow \infty$. Hence, $(1 + \varepsilon)^n > n^m k^{-m}$ for sufficiently large n. Then $n^m k^{n-m} < (1 + \varepsilon)^n k^n = \theta^n$. So if $\sigma \in T$, then $\mu_n(\sim\sigma) \rightarrow_n 0$ geometrically fast.

Now assume that τ is a first-order $\mathscr{S}$-sentence such that $T \models \tau$. By the compactness theorem, there is a finite subset $\{\sigma_1, \cdots, \sigma_r\} \subseteq T$ such that $\models((\sigma_1 \wedge \cdots \wedge \sigma_r) \rightarrow \tau)$, and hence $\models(\sim\tau \rightarrow (\sim\sigma_1 \vee \cdots \vee \sim\sigma_r))$. So $\mu_n(\sim\tau) \leq \mu_n(\sim\sigma_1) + \cdots + \mu_n(\sim\sigma_r)$. It is easy to see that since $\mu_n(\sim\sigma_i) \rightarrow_n 0$ geometrically fast for each i, so does $\mu_n(\sim\sigma_1) + \cdots + \mu_n(\sim\sigma_r)$, and hence so does $\mu_n(\sim\tau)$.

Now let σ be any first-order $\mathscr{S}$-sentence. Either $T \models \sigma$ or $T \models \sim\sigma$. If $T \models \sim\sigma$ then, by what we just showed, $\mu_n(\sigma)$ converges to 0 geometrically fast. If $T \models \sigma$, then $|1 - \mu_n(\sigma)| = \mu_n(\sim\sigma)$, which converges geometrically fast.

We will now show that $\nu_n^{\mathscr{S}}(\sigma)$ converges geometrically fast if $\mathscr{S}$ is not monadic. It is sufficient to assume that $\mu_n(\sigma) \rightarrow_n 1$; if not, then we consider $\sim\sigma$. It is shown in [3] that $a_n(\mathscr{S})/(b_n(\mathscr{S})n!)$ converges to 1 geometrically fast. Since $\mu_n(\sigma)$ converges to 1 geometrically fast, it follows fairly easily from (3) that $\nu_n(\sigma)$ converges to 1 geometrically fast. □

4. Assume that $\mathscr{S} = \{U\}$, where U is a unary predicate symbol. Then $b_n(\mathscr{S}) = n + 1$, since $\mathfrak{A}$ in $\mathscr{A}_n(\mathscr{S})$ is characterized up to isomorphism by the cardinality of the interpretation of U in $\mathfrak{A}$.

Let σ be $\forall x Ux$. Then $\nu_n(\sigma) = 1/(n + 1)$. So $\nu_n(\sigma) \rightarrow_n 0$, but not geometrically fast. This contrasts interestingly with the case $\mathscr{S} = \{P\}$, P a binary predicate symbol, in which $\nu_n^{\mathscr{S}}(\forall xPxx) \rightarrow_n 0$ geometrically fast.

In general, if $\mathscr{S}$ is monadic and σ is a first-order $\mathscr{S}$-sentence, then we can show, as in the proof of Case 1 of Theorem 9, that there are polynomials p and q such that $\nu_n^{\mathscr{S}}(\sigma) = p(n)/q(n)$.

5. If σ and τ are sentences, and if $\mu_n(\tau) > 0$ for each n, then does the conditional probability $\mu_n(\sigma \mid \tau) = \mu_n(\sigma \wedge \tau)/\mu_n(\tau)$ converge? Let σ_1 be $\forall x\exists! y(x \neq y \wedge Pxy \wedge Pyx)$, as in part 1 of this section, and let σ_2 be $\exists w(\forall x \neq w)(\exists! y \neq w)(x \neq y \wedge Pxy \wedge Pyx)$. Then the spectrum of σ_1 (σ_2) is the set of even (odd) positive integers. Let $\sigma = \sigma_1$, and let $\tau = \sigma_1 \vee \sigma_2$. Then

$$\mu_n(\sigma \mid \tau) = \nu_n(\sigma \mid \tau) = \begin{cases} 0, & n \text{ odd}, \\ 1, & n \text{ even}. \end{cases}$$

So $\mu_n(\sigma \mid \tau)$ and $\nu_n(\sigma \mid \tau)$ do not converge.

6. For certain sentences τ, it is true that $\mu_n(\sigma \mid \tau)$ converges for every first-order sentence σ. We will give three examples:

$$\tau_1 = \forall x \sim Pxx \wedge \forall x\forall y(Pxy \leftrightarrow Pyx),$$
$$\tau_2 = \forall x\forall y(x \neq y \rightarrow (Pxy \leftrightarrow \sim Pyx)) \wedge \forall x \sim Pxx,$$
$$\tau_3 = \forall x\forall y(Ux \leftrightarrow Uy).$$

Let $\mathscr{S} = \{P\}$, P a binary predicate symbol, and let τ_1 be as above. Let Q be a graph predicate symbol, that is, a symbol whose interpretation in a structure is a set of unordered pairs. For each $\{P\}$-sentence σ, let σ' be the result of replacing every occurrence of P in σ by Q. Then $\mu_n^{\{P\}}(\sigma \mid \tau_1) = \mu_n^{\{Q\}}(\sigma')$. Modify the set of sentences T of §2 to be a new set T', by a redefinition of a complete open description to be consistent with Q being a graph predicate symbol. Then T' is consistent and complete, and the old proof goes through to show that $\mu_n^{\{Q\}}(\sigma')$ converges to 0 or 1, depending on whether $T' \models \sigma'$ or $T' \models \sim\sigma'$.

The case when τ_2 is as above is similar to the previous case. Then we are effectively dealing with what graph-theorists might call a "tournament" predicate symbol, and again, $\mu_n(\sigma \mid \tau_2)$ converges to 0 or 1.

Let $\mathscr{S} = \{U\}$, U a unary predicate symbol, and let τ_3 be as above. Let $\mu_n'(\sigma) = \mu_n(\sigma \mid \tau_3)$. Then $\mu_n'(\forall xUx) = \frac{1}{2}$, since if $\mathfrak{A} \models \tau_3$, then either $\mathfrak{A} \models \forall xUx$ or $\mathfrak{A} \models \forall x \sim Ux$. And, for any first-order $\mathscr{S}$-sentence σ, the sequence $\langle \mu_n'(\sigma) : n \in Z^+ \rangle$ is eventually $0, \frac{1}{2}$, or 1. For, $\sigma \wedge \forall xUx$ is equivalent to some sentence $\sigma_1 \wedge \forall xUx$, where σ_1 involves only equality. So, for some N_1, we know that either $\sigma \wedge \forall xUx$ has no models of cardinality n for each $n > N_1$, or it has a model of cardinality n for each $n > N_1$. Similarly, find N_2 such that $\sigma \wedge \forall x \sim Ux$ has either no models of cardinality n for each $n > N_2$, or a model of cardinality n for each $n > N_2$. Let $N = \max(N_1, N_2)$. Then, for each $n > N$,

$\mu_n'(\sigma) = 0$, if $\sigma \wedge \forall xUx$ and $\sigma \wedge \forall x \sim Ux$ each have no models of cardinality n;
$\mu_n'(\sigma) = \frac{1}{2}$, if $\sigma \wedge \forall xUx$ has a model of cardinality n but $\sigma \wedge \forall x \sim Ux$ has no model of cardinality n, or if $\sigma \wedge \forall x \sim Ux$ has a model of cardinality n but $\sigma \wedge \forall xUx$ has no model of cardinality n;
$\mu_n'(\sigma) = 1$ otherwise.

So $\mu_n(\sigma \mid \tau_3)$ is eventually $0, \frac{1}{2}$, or 1.

SOME OPEN PROBLEMS. Characterize those sentences τ such that $\mu_n(\sigma \mid \tau)$ converges for each σ. Is there always convergence if τ is universal? What are the possible limits?

7. We will now briefly consider more general probability measures than μ_n and and ν_n. Let $\mathscr{S} = \{P\}$, P a binary predicate symbol, and let $\mathscr{S}' = \mathscr{S} \cup \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \cdots\}$ as before. Let p and α be given, with $0 < p < 1$ and $0 \leq \alpha < 1$. We will define a new probability measure $\mu_n^{\mathscr{S},\alpha}$ on $\mathscr{A}_n(\mathscr{S})$ in which all structures need not be given the same weight, which has the property that $\mu_n^{\mathscr{S},\alpha}(\sigma)$ converges for each first-order $\mathscr{S}$-sentence σ and $\mu_n^{\mathscr{S},\alpha}(\forall x \forall y Pxy)$ converges to a nonzero value.

Let f be a function from the reals to the reals defined by $f(x) = (1 - \alpha)p^x + \alpha$ for each x. Given $\mathfrak{A} = \langle\{1, \cdots, n\}; Q\rangle$ in $\mathscr{A}_n(\mathscr{S})$, where Q is the interpretation of P in $\mathfrak{A}$, assume that Qij $(\sim Qij)$ holds for r (s) different pairs $\langle i, j\rangle$ with $1 \leq i \leq n$, $1 \leq j \leq n$; thus $r + s = n^2$. Then let the weight of $\mathfrak{A}$ be $f(n^2)$, if $s = 0$, and $(1 - f(1))p^r(1 - p)^{s-1}$, if $s > 0$. So, if $p = \frac{1}{2}$ and $\alpha = 0$, then $\mu_n^{\mathscr{S},\alpha} = \mu_n^{\mathscr{S}}$. The measure $\mu_n^{\mathscr{S},\alpha}$ is such that if σ is a conjunction of r distinct sentences $\boldsymbol{Pij}$ and s distinct sentences $\sim\boldsymbol{Pij}$, with no argument $\langle i, j\rangle$ appearing twice, then we can show that

$$\mu_n^{\mathscr{S},\alpha}(\sigma) = \begin{cases} f(r) & \text{if } s = 0, \\ (1 - f(1))p^r(1 - p)^{s-1} & \text{if } s > 0. \end{cases}$$

We state without proof that it is possible to show that $\mu_n^{\mathscr{S},\alpha}(\sigma)$ converges for each first-order $\mathscr{S}$-sentence σ, and that it converges to one of the four values α, $1 - \alpha$, 0, or 1. And, $\mu_n^{\mathscr{S},\alpha}(\forall x \forall y Pxy) \to_n \alpha$. We also remark that $\mathscr{S}$ can be generalized to be any finite set of predicate symbols.

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