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CONTRIBUTIONS TO THE MODEL THEORY OF FINITE STRUCTURES

Abstract

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This thesis is concerned with three topics involving the model theory of finite structures. First, we study the probability that a first-order sentence is true in finite structures. Second, we investigate spectra and generalized spectra, including questions involving the degree of the extra predicate symbols. Third, we analyze and exploit the interrelationship between generalized spectra and automata, to obtain results in both fields.

We begin with our results in probability theory. Let $\mathcal{S}$ be a finite set of (nonlogical) predicate symbols. By an $\mathcal{S}$-structure, we mean a relational structure appropriate for $\mathcal{S}$. Let $\mathcal{A}_n(\mathcal{S})$ be the set of all $\mathcal{S}$-structures with universe $\{1, \ldots, n\}$. For each first-order $\mathcal{S}$-sentence σ, let $\mu_n(\sigma)$ be the fraction of members of $\mathcal{A}_n(\mathcal{S})$ for which σ is true. We show that $\mu_n(\sigma)$ always converges to 0 or 1, as $n \to \infty$. Let $\mathcal{B}_n(\mathcal{S})$ be a subset of $\mathcal{A}_n(\mathcal{S})$ which contains for each $\mathfrak{A}$ in $\mathcal{A}_n(\mathcal{S})$ exactly one structure isomorphic to $\mathfrak{A}$. For each first-order $\mathcal{S}$-sentence σ, let $\nu_n(\sigma)$ be the fraction of members of $\mathcal{B}_n(\mathcal{S})$ for which σ is true. We show that if all predicate symbols in $\mathcal{S}$ are of the same degree, then $\nu_n(\sigma)$ converges, and $\lim \mu_n(\sigma) = \lim \nu_n(\sigma)$.

Now we turn to the second topic. Let σ be a first-order sentence, and let $P_1, \ldots, P_m$ be those (nonlogical) predicate symbols in σ which are not in $\mathcal{S}$ (these are the <u>extra predicate symbols</u>.) Let σ' be the existential second-order sentence $\exists P_1 \ldots \exists P_m \sigma$. The <u>$\mathcal{S}$-spectrum</u> (or <u>generalized spectrum</u>) of σ' is the class of finite $\mathcal{S}$-structures in which σ' is true. When $\mathcal{S} = \emptyset$, we can identify the $\mathcal{S}$-spectrum of σ' with the set of cardinalities of finite structures in which σ is true. This set, called the <u>spectrum</u> of σ, was first considered by H. Scholz (J. of Symbolic Logic 17, 160).

It is natural to ask whether the following two statements are true:

(S) The complement of every spectrum is a spectrum.

(GS) The complement of every generalized spectrum is a generalized spectrum.

The question of the truth of (S) was first raised by G. Asser in 1955 (Zeit. für math. Logik und Grundlagen d. Math. 1, 252-263). The second statement implies the first, and the truth of neither is known. A large portion of this thesis is concerned with problems related to these two statements.

Define $\mathfrak{F}_k(\mathcal{S})$ to be the class of those $\mathcal{S}$-spectra where all the extra predicate symbols are k-ary, and call every $\mathcal{S}$-spectrum in $\mathfrak{F}_1(\mathcal{S})$ <u>monadic</u>. We introduce a Fraïssé-type game to show that the class of monadic generalized spectra is not closed under complement.

We show that for each spectrum A there is a positive integer k such that $\{n^k: n \in A\}$ is a spectrum involving only one binary predicate symbol. We use this as a tool for showing that if there is a counterexample to various conjectures about spectra, then there is a spectrum involving only one binary predicate symbol which is a counterexample.

We show that there is an exact trade-off between the degree of the extra predicate symbols and the cardinality of an "extra universe." In the case of spectra, we find that if A is a set of positive integers, and if $k \geq 2$, then A is in $\mathcal{J}_{k+1}(\emptyset)$ iff $\{n[n^{1/k}]: n \in A\}$ is in $\mathcal{J}_k(\emptyset)$, where $[x]$ is the greatest integer not exceeding x. We use the trade-off to show that if $p \geq 2$ and $\mathcal{J}_p(\mathcal{S}) = \mathcal{J}_{p+1}(\mathcal{S})$, then $\mathcal{J}_k(\mathcal{S}) = \mathcal{J}_p(\mathcal{S})$ for each $k \geq p$.

Another result is a "two-cardinal" characterization of spectra whose complements are spectra.

Now we turn to the interrelationship between generalized spectra and automata. Consider the following two statements of automata theory:

(S_1) The class of sets recognizable by a nondeterministic Turing machine in exponential time is closed under complement.

(GS_1) The class of sets recognizable by a non-deterministic Turing machine in polynomial time is closed under complement.

We show that (GS) and (GS_1) are equivalent. This supplements the known result that (S) and (S_1) are equivalent (This latter result is apparently due to James Bennett, but was never published; it was also proved by Jones and Selman in Proc. 4th ACM Symposium on Thy. of Computing, 157-167.)

Using automata theory, we obtain among others the following results about (generalized) spectra: we find certain specific generalized spectra $\mathcal{Q}$, including a monadic generalized spectrum, such that (GS) holds iff the complement of $\mathcal{Q}$ is a generalized spectrum; we find a specific spectrum A involving only one binary predicate symbol such that (S) holds iff the complement of A is a spectrum; we find a spectrum A involving only one binary predicate symbol, such that $\{n: 2^n \in A\}$ is not a spectrum; and we show that for many spectra A, we can find a sentence σ which has at most one model of each finite cardinality, such that A is the spectrum of σ.

We also prove various results in automata theory itself. For example, we show that if the classes of sets which are polynomial-time recognizable by deterministic and nondeterministic Turing machines are the same, then the following apparently much stronger condition holds: there is a constant k such that essentially any set that can be recognized nondeterministically in time T can be recognized deterministically in time T^k.

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