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A TWO-CARDINAL CHARACTERIZATION OF DOUBLE SPECTRA

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§ 1. Introduction

Let σ be a sentence of first-order logic (with equality). SCHOLZ [3] defined the *spectrum* of σ to be the set of cardinalities of finite structures in which σ is true. ASSER [1] asked whether the complement of every spectrum is a spectrum. This question is still open. Call a set S of positive integers a *double spectrum* if both S and the complement $\tilde{S}$ of S are spectra. In this paper, we give a "two-cardinal" characterization of double spectra.

§ 2. Definitions

If A is a set, then denote the cardinality of A by card(A). Denote the set $\{0, 1, 2, \ldots\}$ of natural numbers by N, and the set $\{1, 2, 3, \ldots\}$ of positive integers by Z^+.

Let $\mathscr{S}$ be a set of (nonlogical) predicate symbols. By an *$\mathscr{S}$-structure,* we mean a relational structure appropriate for $\mathscr{S}$. If $\mathfrak{A}$ is an $\mathscr{S}$-structure, then denote the cardinality of the universe of $\mathfrak{A}$ by card($\mathfrak{A}$). If card($\mathfrak{A}$) is finite, then we call $\mathfrak{A}$ a finite ($\mathscr{S}$)-structure. If $P \in \mathscr{S}$, then by $P^{\mathfrak{A}}$, we mean the interpretation of P in $\mathfrak{A}$.

If $\mathfrak{A}$ is a structure and σ is a first-order sentence, then by $\mathfrak{A} \models \sigma$, we mean that σ is true in $\mathfrak{A}$.

For ease in readability, we may abbreviate a first-order sentence by its English translation, in quotation marks.

Let φ be a first-order formula with free variables $x, v_1, \ldots, v_m$, where we single out the free variable x. We will define the *relativization* ψ^φ for first-order formulas ψ, by induction on formulas. If ψ is atomic, then $\psi^\varphi = \psi$; in addition,

$$(\sim\psi)^\varphi = \sim(\psi^\varphi), \quad (\psi_1 \wedge \psi_2)^\varphi = \psi_1^\varphi \wedge \psi_2^\varphi, \quad (\forall y\psi)^\varphi = \forall z(\varphi(z, v_1, \ldots, v_m) \to \psi(z)),$$

where $\varphi(z, v_1, \ldots, v_m)$ (respectively, $\psi(z)$) is the result of replacing each occurrence of x in φ (respectively, y in ψ) by a new variable z, chosen by some fixed rule. If U is a unary predicate symbol, then ψ^U denotes ψ^{Ux}.

§ 3. A Two-Cardinal Characterization

If S is a set of positive integers, then let $f_S\colon \mathsf{Z}^+ \to \mathsf{N}$ be the function which maps n onto the cardinality of $S \cap \{1, \ldots, n\}$, for each n.

Let U be a unary predicate symbol, and let $g\colon \mathsf{Z}^+ \to \mathsf{N}$ have the property that $g(n) \leqq n$ for each n. Then we say that g is *two-cardinal definable* (*via* σ) if σ is a first-order sentence which contains, among others, the symbol U, and

1. If $\mathfrak{A}$ is a finite structure and $\mathfrak{A} \models \sigma$, then $g(\mathrm{card}(\mathfrak{A})) = \mathrm{card}(U^{\mathfrak{A}})$.
2. If $g(n) = u$, then there is a (finite) structure $\mathfrak{A}$ such that $\mathrm{card}(\mathfrak{A}) = n$, $\mathrm{card}(U^{\mathfrak{A}}) = u$, and $\mathfrak{A} \models \sigma$.

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Theorem. *S is a double spectrum iff f_S is two-cardinal definable.*

Proof. "$\Leftarrow$": Assume that f_S is two-cardinal definable via σ. For each k and each k-ary predicate symbol P which appears in σ, let $\hat{P}$ be a new k-ary predicate symbol. Let V be one more new unary predicate symbol. Let τ be the sentence

$$\text{"There is exactly one point not in } V\text{"} \wedge \text{"}\hat{U} \text{ is a proper subset of } U\text{"}.$$

Form $\hat{\sigma}$ from σ by replacing each symbol P by $\hat{P}$, for each P. If $n \neq 1$ and if $f_S(n) \neq 0$, then $n \in S$ iff n is in the spectrum of the sentence $\sigma \wedge \tau \wedge \hat{\sigma}^V$. This is because $n \in S$ iff $f_S(n) > f_S(n-1)$. So S differs from a spectrum by at most a finite set, and hence is a spectrum itself. If $n \neq 1$, then $n \in \bar{S}$ iff n is in the spectrum of

$$\sigma \wedge \text{"There is exactly one point not in } V\text{"} \wedge \text{"}\hat{U} = U\text{"} \wedge \hat{\sigma}^V.$$

This is because $n \in \bar{S}$ iff $f_s(n) = f_S(n-1)$. So $\bar{S}$ is a spectrum.

"$\Rightarrow$": Assume that S is the spectrum of σ, and $\bar{S}$ is the spectrum of τ. We can assume that σ and τ have no nonlogical symbols in common. For each k and each k-ary predicate symbol P which occurs in σ or τ, let $\hat{P}$ be a new $(k+1)$-ary predicate symbol. Let U be a new unary predicate symbol, and $<$ a new binary predicate symbol. Let y be a variable that does not occur in σ or τ, and let $\varphi(x)$ be the formula $x < y \vee x = y$. Define $\hat{\sigma}(y)$ to be the formula obtained from σ by replacing each occurrence of $Px_1 \ldots x_k$ in σ by $\hat{P}yx_1 \ldots x_k$, for each P in σ and each variable $x_1, \ldots, x_k$; similarly, define $\hat{\tau}(y)$. Let α be the following sentence:

$$\text{"} < \text{is a strict linear order (transitive and satisfies trichotomy)"} \wedge$$
$$\wedge \forall y((\hat{\sigma}^{\varphi(x)} \vee \hat{\tau}^{\varphi(x)}) \wedge (Uy \leftrightarrow \hat{\sigma}^{\varphi(x)})).$$

Intuitively, α says that for each initial segment $A_y = \{x : x \leqq y\}$, we have imposed new relations on A_y, such that a simulated version of either σ or τ is true about this new structure $\mathfrak{A}_y$ with universe A_y; the relation $Px_1 \ldots x_k$ is simulated by $\hat{P}yx_1 \ldots x_k$. Further, α says that Uy holds iff the simulated version of σ is true about $\mathfrak{A}_y$. Therefore, it is not hard to see that f_S is two-cardinal definable via α.

§ 4. An Example

Let $f(n) = [n^{1/2}]$ for each positive integer n (where $[x]$ is the greatest integer not exceeding x). Then $f = f_S$, where S is the set of perfect squares: this is clear upon reflection. So f is two-cardinal definable iff S and $\bar{S}$ are each spectra (which, it is not hard to show, is the case).

Bibliography

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