Zeitschr. f. math. Logik und Grundlagen d. Math.
Bd. 21, S. 123–134 (1975)

A SPECTRUM HIERARCHY

by RONALD FAGIN in Yorktown Heights, N.Y. (U.S.A.)[1])

1. Introduction

Let $\mathscr{S}$ be a finite similarity type, that is, a finite set of (nonlogical) predicate symbols. By an *$\mathscr{S}$-structure*, we mean a relational structure suitable for $\mathscr{S}$. Let σ be a first-order sentence (with equality), and let $P_1, \ldots, P_m$ be those (nonlogical) predicate symbols in σ which are not in $\mathscr{S}$ (these are the *extra* predicate symbols). Let σ' be the existential second-order sentence $\exists P_1 \ldots \exists P_m \sigma$. The *$\mathscr{S}$-spectrum* (or *generalized spectrum*) of σ' is the class of finite $\mathscr{S}$-structures in which σ' is true. This corresponds to TARSKI's [7] notion of PC, where we restrict our attention to the class of finite structures. When $\mathscr{S} = \emptyset$, we can identify the $\mathscr{S}$-spectrum of σ' with the set of cardinalities of finite structures in which σ is true. This set, called the *spectrum* of σ, was first considered by H. SCHOLZ [6].

We show that for each spectrum A, there is a positive integer k such that $\{n^k : n \in A\}$ is a spectrum involving only one binary predicate symbol. We use this to show that if there are spectra with certain properties, then there are spectra involving only one binary predicate symbol which have those properties.

Define $\mathscr{F}_k(\mathscr{S})$ to be the class of those $\mathscr{S}$-spectra in which all of the extra predicate symbols are k-ary. We show that there is an exact trade-off between the degree of the extra predicate symbols and the cardinality of an "extra universe". In the case of spectra, we find that if A is a set of positive integers, and if $k \geqq 2$, then A is in $\mathscr{F}_{k+1}(\emptyset)$ iff $\{n[n^{1/k}] : n \in A\}$ is in $\mathscr{F}_k(\emptyset)$, where $[x]$ is the greatest integer not exceeding x. We use the trade-off to show that if $\mathscr{F}_p(\mathscr{S}) = \mathscr{F}_{p+1}(\mathscr{S})$, then $\mathscr{F}_k(\mathscr{S}) = \mathscr{F}_p(\mathscr{S})$ for each $k \geqq p$. It is an open problem as to whether there is any spectrum not in $\mathscr{F}_2(\mathscr{S})$, or, indeed, whether there is any $\mathscr{S}$-spectrum not obtainable by using only one extra binary predicate symbol.

2. Definitions

Denote the set of positive integers $\{1, 2, 3, \ldots\}$ by Z^+, and the set $\{0, \ldots, n-1\}$ by n. If A is a set, then $\bar{A}$ is the cardinality of the set. Denote the set of k-tuples $\langle a_1, \ldots, a_k \rangle$ of members of A by A^k.

If $\mathscr{S}$ is a finite similarity type and $\mathfrak{A}$ is an $\mathscr{S}$-structure (both defined earlier), then we denote the universe of $\mathfrak{A}$ by $|\mathfrak{A}|$, the cardinality of $|\mathfrak{A}|$ by card($\mathfrak{A}$), and the interpretation (in $\mathfrak{A}$) of P in $\mathfrak{A}$ by $P^{\mathfrak{A}}$. If card($\mathfrak{A}$) is finite, then we call $\mathfrak{A}$ a *finite $\mathscr{S}$-structure*. Denote the class of finite $\mathscr{S}$-structures by *Fin*($\mathscr{S}$).

In addition to the usual types of predicate symbols (unary, binary, etc.), we will allow a special type of predicate symbol, a *graph predicate symbol*. If $P \in \mathscr{S}$ and P is a graph predicate symbol, then for $\mathfrak{A}$ to be an $\mathscr{S}$-structure, $P^{\mathfrak{A}}$ must be a graph (i.e., irreflexive and symmetric), or, equivalently, a set of unordered pairs of members of $|\mathfrak{A}|$.

---

[1]) This paper is based on a part of the author's doctoral dissertation [2] in the Department of Mathematics at the University of California, Berkeley. Part of this work was carried out while the author was a National Science Foundation Graduate Fellow; also, part of this work was supported by NSF Grant No. GP-24532.

The author is grateful to ROBERT VAUGHT and WILLIAM CRAIG for useful suggestions which improved readability.

Assume that $\mathscr{S}$ and $\mathscr{T}$ are disjoint finite similarity types, that $\mathfrak{A}$ is an $\mathscr{S} \cup \mathscr{T}$-structure, and that $\mathfrak{B}$ is an $\mathscr{S}$-structure. Then $\mathfrak{A}$ is an *expansion* of $\mathfrak{B}$ (to $\mathscr{S} \cup \mathscr{T}$), written $\mathfrak{B} = \mathfrak{A} \upharpoonright \mathscr{S}$, if $|\mathfrak{A}| = |\mathfrak{B}|$, and $P^{\mathfrak{A}} = P^{\mathfrak{B}}$ for each P in $\mathscr{S}$.

Assume that $\mathfrak{A}$ and $\mathfrak{B}$ are $\mathscr{S}$-structures (and, for convenience, that $\mathscr{S}$ contains no graph predicate symbols). Then $\mathfrak{B}$ is a *substructure* of $\mathfrak{A}$ (written $\mathfrak{B} \subseteq \mathfrak{A}$), if $|\mathfrak{B}| \subseteq |\mathfrak{A}|$, and $P^{\mathfrak{B}} = P^{\mathfrak{A}} \cap |\mathfrak{B}|^k$ for each k-ary predicate symbol P in $\mathscr{S}$.

If $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic $\mathscr{S}$-structures, then we write $\mathfrak{A} \cong \mathfrak{B}$. If $\mathscr{A}$ is a class of structures, then by Isom($\mathscr{A}$), we mean the closure of $\mathscr{A}$ under isomorphism, that is, $\{\mathfrak{B}: (\exists \mathfrak{A} \in \mathscr{A})\ (\mathfrak{A} \cong \mathfrak{B})\}$.

Let φ be a first-order formula. If each nonlogical symbol appearing in φ is in $\mathscr{S}$, then we call φ an $\mathscr{S}$-formula. If $\mathscr{T} = \{P_1, \ldots, P_m\}$, then by $\exists \mathscr{T} \varphi$, we mean the existential second-order formula $\exists P_1 \ldots \exists P_m \varphi$. The formula $\exists ! x \varphi$ (read: "there is exactly one x such that φ") is defined as usual. For ease in readability, we will often abbreviate first-order formulas by their English equivalents, in quotation marks. If Γ is a finite set of formulas, then by $\bigwedge \{\varphi: \varphi \in \Gamma\}$, we mean the conjunction of the formulas of Γ; similarly for $\bigvee \{\varphi: \varphi \in \Gamma\}$.

If $x_1, \ldots, x_m$ are (individual) variables, then we will sometimes write $\boldsymbol{x}$ as an abbreviation for the m-tuple $\langle x_1, \ldots, x_m \rangle$, when this will lead to no confusion. We may write $\forall \boldsymbol{x} \varphi$ for $\forall x_1 \ldots \forall x_m \varphi$.

Let φ be a first-order formula with free variables $x, v_1, \ldots, v_m$, where we single out the free variable x. We will define the *relativization* ψ^{φ} for first-order formulas ψ, by induction on formulas. If ψ is atomic, then $\psi^{\varphi} = \psi$; in addition,

$$(\sim \psi)^{\varphi} = \sim(\psi^{\varphi}), \quad (\psi_1 \wedge \psi_2)^{\varphi} = \psi_1^{\varphi} \wedge \psi_2^{\varphi}, \quad (\forall y \psi)^{\varphi} = \forall z(\varphi(z, v_1, \ldots, v_m) \rightarrow \psi(z)),$$

where $\varphi(z, v_1, \ldots, v_m)$ (respectively, $\psi(z)$) is the result of replacing each occurrence of x in φ (respectively, y in ψ) by a new variable z, chosen by some fixed rule.

Let $\mathscr{S}$ be fixed, and let σ be a sentence with all of its (nonlogical) predicate symbols in $\mathscr{S}$. If σ is true in $\mathfrak{A}$, then we write $\mathfrak{A} \models \sigma$, and we say that $\mathfrak{A}$ is a model of σ. By $Mod_{\omega}\, \sigma$, we mean the class of all finite $\mathscr{S}$-structures which are models of σ.

Define $\mathscr{F}_k(\mathscr{S})$ as before. Let BIN($\mathscr{S}$) be the class of all $\mathscr{S}$-spectra involving only one extra graph predicate symbol. Thus, if $\mathscr{A} = Mod_{\omega}\, \exists P \sigma$, where P is a graph predicate symbol, where σ is first-order, and where $\mathscr{S}$ contains every nonlogical symbol in σ except P, then $\mathscr{A} \in$ BIN($\mathscr{S}$). Obviously, BIN($\mathscr{S}$) $\subseteq \mathscr{F}_2(\mathscr{S})$. We abbreviate $\mathscr{F}_k(\emptyset)$ and BIN($\emptyset$) by $\mathscr{F}_k$ and BIN.

3. A reduction

In this section, we show (Theorem 3) that for each spectrum S there is a positive integer k such that $\{n^k: n \in S\}$ is in BIN. Theorem 3 is a useful tool for showing that if there is a counterexample to certain conjectures about spectra, then a counterexample occurs in BIN, the lowest interesting level of the spectrum hierarchy (it is well-known that by an elimination-of-quantifiers argument, $\mathscr{F}_1$ can be shown to contain only finite and cofinite sets).

In 1955, Asser [1] posed the question of whether the complement of every spectrum is a spectrum. We will show that it follows from Theorem 3 that if there is a spectrum whose complement is not a spectrum, then there is such a spectrum in BIN. Actually, this result is improved by the author in [2] and [3], where Theorem 3 is used to find a *particular* spectrum in BIN whose complement is a spectrum iff the complement of every spectrum is a spectrum. Similarly, from the result in [2] and [3] that there is a spectrum S such that $\{n: 2^n \in S\}$ is not a spectrum, it is shown there that there is such a spectrum S in BIN.

If $\mathscr{A}$ and $\mathscr{B}$ are subsets of $Fin(\mathscr{S})$, then we say that *$\mathscr{B}$ is a finite modification of $\mathscr{A}$* if for some constant N, whenever $\mathfrak{A} \in Fin(\mathscr{S})$ and $\text{card}(\mathfrak{A}) \geqq N$, then $\mathfrak{A} \in \mathscr{A}$ iff $\mathfrak{A} \in \mathscr{B}$. The following well-known simple lemma is very useful.

Lemma 1. *If $\mathscr{A}$ is an $\mathscr{S}$-spectrum and $\mathscr{B}$ is a finite modification of $\mathscr{A}$, then $\mathscr{B}$ is an $\mathscr{S}$-spectrum with the same extra predicate symbols.*

Proof. It is well-known that for each $\mathfrak{A}$ in $Fin(\mathscr{S})$, there is a first-order $\mathscr{S}$-sentence σ, such that if $\mathfrak{B} \in Fin(\mathscr{S})$, then $\mathfrak{B} \cong \mathfrak{A}$ iff $\mathfrak{B} \models \sigma$. The lemma now follows easily.

We will now begin our analysis of the hierarchy $\langle \mathscr{F}_k(\mathscr{S}): k \in \mathsf{Z}^+ \rangle$. If $\mathscr{A}$ is an $\mathscr{S}$-spectrum, then there is k such that $\mathscr{A} \in \mathscr{F}_k(\mathscr{S})$: this follows from well-known techniques of simulating $(k-1)$-ary relations by k-ary relations.

Let $\mathfrak{A} = \langle A; Q_1, \ldots, Q_m \rangle$ be a finite $\mathscr{S}$-structure, and let $f: \mathsf{Z}^+ \to \mathsf{Z}^+$ be a function with $f(n) \geqq n$ for each n. We will define a new $\mathscr{S}$-structure $\bar{f}(\mathfrak{A})$ as follows. Let B be a set of cardinality $f(\bar{A})$, such that $A \subseteq B$; for definiteness, we could say that $B = A \cup C$, where C is the set of the first $f(\bar{A}) - \bar{A}$ positive integers which are not in A. Then we let $\bar{f}(\mathfrak{A}) = \langle B; Q_1, \ldots, Q_m \rangle$. The structure $\bar{f}(\mathfrak{A})$ can be thought of as the structure $\mathfrak{A}$, along with an extra universe of cardinality $f(\text{card}(\mathfrak{A})) - \text{card}(\mathfrak{A})$. The old universe is not named.

The following lemma is straightforward to prove.

Lemma 2. *Assume that $f: \mathsf{Z}^+ \to \mathsf{Z}^+$ is a one-one function with $f(n) \geqq n$ for each n. Then $\bar{f}: Fin(\mathscr{S}) \to Fin(\mathscr{S})$ is essentially one-one, in the sense that if $\bar{f}(\mathfrak{A}) \cong \bar{f}(\mathfrak{B})$, then $\mathfrak{A} \cong \mathfrak{B}$.*

We will now prove that if $k \geqq 2$ and $S \in \mathscr{F}_k$, then $\{n^k: n \in S\}$ is in BIN. It will be no more work to prove the generalization to $\mathscr{S}$-spectra; we will utilize the generalization later.

Theorem 3. *Let $f_k: \mathsf{Z}^+ \to \mathsf{Z}^+$ be the function $n \mapsto n^k$. Assume that $\mathscr{A} \in \mathscr{F}_k(\mathscr{S})$, with $k \geqq 2$. Then* $\text{Isom}(\{\bar{f}_k(\mathfrak{A}): \mathfrak{A} \in \mathscr{A}\})$ *is in* $\text{BIN}(\mathscr{S})$.

Proof. Assume that $\mathscr{A} = Mod_\omega \exists \mathscr{T} \sigma$, where $\mathscr{T}$ is a set of t distinct k-ary predicate symbols $P_1, \ldots, P_t$. Let P be a graph predicate symbol. We will now write $\{P\}$-formulas $\psi_1, \ldots, \psi_{2k}$ that define $2k$ "levels" $L_1, \ldots, L_{2k}$ which partition the universe into $2k$ sets. Let r be the least integer exceeding $\log_2(2k-1)$. Let ψ_1 be the formula $\bigvee_{i=1}^{r} (v_0 = v_i)$, where $v_0, \ldots, v_r$ are distinct variables; $\psi_1(v_0)$ defines the level L_1. The levels $L_2, \ldots, L_{2k}$ are defined by stating how their points relate (via P) to the points in L_1. Specifically, let Φ be the set of 2^r formulas $\sim\psi_1 \wedge \bigwedge_{i=1}^{r} \gamma_i$, where γ_i can be either Pv_0v_i or $\sim Pv_0v_i$ $(1 \leqq i \leqq r)$. The set Φ contains at least $2k-1$ distinct formulas, since $r \geqq \log_2(2k-1)$. Let $\psi_2, \ldots, \psi_{2k-1}$ be distinct formulas in Φ, and let ψ_{2k} be the disjunction of the remaining formulas in Φ. If x is a variable, then by $\psi_i(x)$, for $1 \leqq i \leqq 2k$, we mean the result of substituting x in ψ_i for v_0 (of course, $v_1, \ldots, v_r$ are also free). As promised, the levels $L_1, \ldots, L_{2k}$ (L_i defined by $\psi_i(x)$ for each i) partition the universe, when the values of $v_1, \ldots, v_r$ are distinct and are held fixed.

We now write a formula τ (with $v_1, \ldots, v_r$ free) which forces the universe to be of cardinality n^k, where n is the cardinality of L_2. Let τ be the conjunction of the following formulas:

"L_1 contains exactly r points".

"There is a one-one correspondence between pairs $\{x, y\}$ of points in L_2 and points z in L_3, where the correspondence is given by $Pxz \wedge Pyz$".

$\bigwedge_{m=2}^{k-1}$ "There is a one-one correspondence between pairs $\{x, y\}$ (where $x \in L_2$ and $y \in L_{2m-1}$), and points z in L_{2m+1}, given by $Pxz \wedge Pyz$".

$\bigwedge_{m=2}^{k-1}$ "There is a one-one correspondence between points x in L_{2m-1} and y in L_{2m}, given by Pxy".

"There is a set X of exactly r points in L_{2k-1}, for which there is a one-one correspondence between points x in $(L_{2k-1} - X)$ and points y in L_{2k}, given by Pxy". Then τ forces L_1 to contain exactly r points, and it forces L_3 to have cardinality $n(n-1)/2$, where n is the cardinality of L_2; we can show inductively on m that τ forces L_{2m+1} and L_{2m+2} to each have cardinality $(n^{m+1} - n^m)/2$, for $1 \leqq m \leqq k-1$, except that L_{2k} is forced to have cardinality $(n^k - n^{k-1})/2 - r$. So altogether, τ forces the universe to have cardinality n^k.

We think of L_2 as constituting a small universe, and we simulate $P_s x_1 \ldots x_k$, where $x_1, \ldots, x_k$ run through L_2, by a $\{P\}$-formula $\varphi_s x_1 \ldots x_k$, $1 \leqq s \leqq t$. The approach is modeled after that of RABIN and SCOTT [5]. Essentially, $\varphi_s x_1 \ldots x_k$ says that for some $y_1, y_2, \ldots$ in L_{2k}, the situation in Figure 1 occurs, where a and b are joined by a line segment if Pab holds (we assume in Figure 1 that $k = 4$).

Specifically, let φ_s be $\exists \boldsymbol{y}\beta_s$, where β_s is the conjunction of the following formulas:

$$\bigwedge_{i=1}^{k} \psi_2 x_i, \quad \bigwedge_{i} \psi_{2k} y_i \wedge \bigwedge_{i,j} \psi_{2k} y_1^{(j)}, \quad \bigwedge_{i=1}^{s+1} P y_i y_{i+1} \wedge P y_{s+2} y_1,$$

$$P y_1 x_1, \quad P y_1 y_1^{(2)} \wedge P y_1^{(2)} x_2, \quad \bigwedge_{a=3}^{k} (P y_1 y_1^{(a)} \wedge \bigwedge_{i=1}^{a-2} P y_i^{(a)} y_{i+1}^{(a)} \wedge P y_{a-1}^{(a)} x_a).$$

If $z_1, \ldots, z_k$ are variables, then by $\varphi_s(z_1, \ldots, z_k)$, we mean the result of substituting z_i for x_i in φ_s, $1 \leqq i \leqq k$ (φ_s also has free variables $v_1, \ldots, v_r$). Let σ' be the result of replacing $P_s z_1 \ldots z_k$ in the relativization $\sigma^{\psi_2(x)}$ by $\varphi_s z_1 \ldots z_k$, for each s and each variable $z_1, \ldots, z_k$.

For each Q in $\mathscr{S}$, if Q is m-ary then let α_Q be the formula $\forall x_1 \ldots \forall x_m (Q x_1 \ldots x_m \rightarrow \rightarrow \bigwedge_{i=1}^{m} \psi_2 x_i)$. Let $\mathscr{B} = Mod_\omega \exists P(\exists v_1 \ldots \exists v_r(\tau \wedge \sigma' \wedge \bigwedge_{Q \in \mathscr{S}} \alpha_Q))$, and let $\mathscr{B}' = \text{Isom}(\{\overline{f_k}(\mathfrak{A}): \mathfrak{A} \in \mathscr{A}\})$. Then, it is fairly straightforward to check that $\mathscr{B}'$ is a finite modification of $\mathscr{B}$. The only difficulty lies in seeing that there are enough points in L_{2k} to carry out the construction typified by Figure 1. Referring to Figure 1 (and assuming that

$k = 4$), we pick a set of new points y (each in L_{2k}) for each *triple* $\langle x_1, x_2, x_3 \rangle$ of points in L_2, where we plan to simulate $P_s x_1 x_2 x_3 x_4$, $P_s x_1 x_2 x_3 x_4'$, and $P_s x_1 x_2 x_3 x_4''$ by the construction of Figure 2.

A simple estimate shows that the number of these extra points y needed is bounded by a polynomial in n of degree $k - 1$; since the number of points in L_{2k} is a polynomial in n of degree k, the construction is possible for sufficiently large n.

Since $\mathscr{B} \in \mathrm{BIN}(\mathscr{S})$ and $\mathscr{B}'$ is a finite modification of $\mathscr{B}$, it follows from Lemma 1 that $\mathscr{B}' \in \mathrm{BIN}(\mathscr{S})$, which was to be shown.

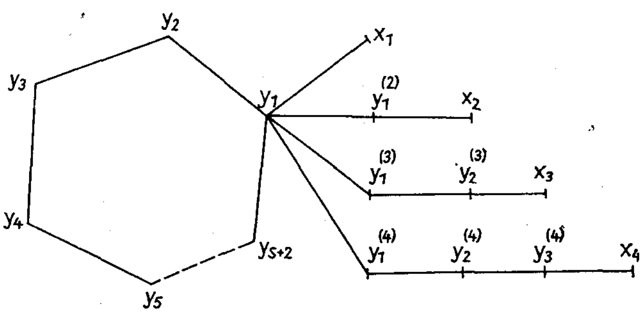

Figure 1

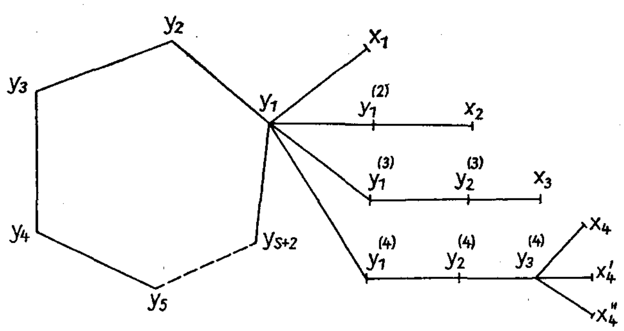

Figure 2

Corollary 4. *Let S be a spectrum. Then $\{n^k : n \in S\}$ is in* BIN, *for some positive integer k.*

Proof. Immediate from Theorem 3.

We remark that although we do not need it, we can strengthen Corollary 4 as follows: Let S be a spectrum. Then there is a positive integer k_0 such that for each positive integer $k \geqq k_0$, the set $\{n^k : n \in S\}$ is in BIN.

We are now almost prepared to prove the following theorem.

Theorem 5. *The complement of every spectrum is a spectrum iff for each A in* BIN, *the set $\tilde{A}$ is a spectrum.*

We need the following lemma.

Lemma 6. *Let k be a positive integer and S a spectrum. Then $\{n : n^k \in S\}$ is a spectrum.*

Proof. Assume that $S = \{n: \langle n\rangle \models \exists\mathscr{T}\sigma\}$, where $\mathscr{T}$ is a finite set of predicate symbols. Let $\mathscr{T}'$ be a new set of predicate symbols obtained from $\mathscr{T}$ by replacing each r-ary predicate symbol Q in $\mathscr{T}$ by an (rk)-ary predicate symbol Q'. Let $X = \{x_i: 1 \leqq i \leqq m\}$ be the set of variables that occur in σ (assume that x_i and x_j are distinct unless $i = j$). Let $\{x_i^j: 1 \leqq i \leqq m,\ 1 \leqq j \leqq k\}$ be another set of variables, where x_i^j and $x_{i'}^{j'}$ are distinct variables unless $i = i'$ and $j = j'$. For each first-order $\mathscr{T}$-formula φ with all variables in X, we define a $\mathscr{T}'$-formula φ^*, by induction on formulas:

$$(x_i = x_j)^* \quad \text{is} \quad \bigwedge_{s=1}^{k} (x_i^s = x_j^s),$$

$$(Qx_{i_1} \ldots x_{i_r})^* \quad \text{is} \quad Q'x_{i_1}^1 \ldots x_{i_1}^k x_{i_2}^1 \ldots x_{i_2}^k \ldots x_{i_r}^1 \ldots x_{i_r}^k,$$

$$(\varphi_1 \wedge \varphi_2)^* \quad \text{is} \quad \varphi_1^* \wedge \varphi_2^*, \qquad (\sim\varphi)^* \quad \text{is} \quad \sim(\varphi^*), \qquad (\forall x_i\varphi)^* \quad \text{is} \quad \forall x_i^1 \ldots \forall x_i^k \varphi^*.$$

It is easy to see that $\{n: n^k \in S\} = \{n: \langle n\rangle \models \exists\mathscr{T}'\sigma^*\}$. The set of k-tuples effectively serve as an "imaginary universe".

We remark that Lemma 6 can be strengthened as follows. Let S be a spectrum, and let p be any polynomial with rational coefficients. Then $\{n: p(n) \in S\}$ is a spectrum.

Proof of Theorem 5.

$\Rightarrow$: Obvious.

$\Leftarrow$: Let S be a spectrum. Find k from Corollary 4 such that $T = \{n^k: n \in S\}$ is in BIN. Since $n \mapsto n^k$ is one-one, it is clear that $\tilde{S} = \{n: n^k \in \tilde{T}\}$. Now $\tilde{T}$ is a spectrum by hypothesis. So by Lemma 6, so is $\tilde{S}$.

We improve Theorem 5 in [2] and [3], by showing that BIN contains a "complete" spectrum (a spectrum S such that $\tilde{S}$ is a spectrum iff the complement of every spectrum is a spectrum); to show this, we again make use of Corollary 4. At this stage, we could prove an analog of Theorem 5 for generalized spectra. However, in [2] and [3] we prove the much stronger result that there is a "complete" generalized spectrum (whose complement is a generalized spectrum iff the complement of every generalized spectrum is a generalized spectrum) which is *monadic*, that is, all of whose extra predicate symbols are unary. This generalized spectrum is $Mod_\omega \exists U \forall x \exists! y(Pxy \wedge Uy)$, where P is a binary predicate symbol and U is a unary predicate symbol. Note that it is too much to hope for a complete (ordinary) spectrum which is monadic, since as remarked earlier, if $A \in \mathscr{F}_1$, then A is either finite or cofinite. We also remark that it is shown in [2] and [4] that there is a monadic generalized spectrum (the class of all nonconnected graphs) whose complement is not a monadic generalized spectrum.

4. A hierarchy of extra predicate symbols

In this section we show that there is an exact trade-off between the degree of the extra predicate symbols and the cardinality of an "extra universe". As a consequence of the trade-off, we show that if $p \geqq 2$ and $\mathscr{F}_p(\mathscr{S}) = \mathscr{F}_{p+1}(\mathscr{S})$, then $\mathscr{F}_k(\mathscr{S}) = \mathscr{F}_p(\mathscr{S})$ for each $k \geqq p$.

Theorem 7. *Let g_k be the function $n \mapsto n[n^{1/k}]$, where $[x]$ is the greatest integer not exceeding x. Assume that $\mathscr{A}$ is a class of $\mathscr{S}$-structures which is closed under isomorphism, and that $k \geqq 2$. Then $\mathscr{A}$ is in $\mathscr{F}_{k+1}(\mathscr{S})$ iff* $\text{Isom}(\{\overline{g_k}(\mathfrak{A}): \mathfrak{A} \in \mathscr{A}\})$ *is in $\mathscr{F}_k(\mathscr{S})$.*

The intuitive idea of the theorem (although it is not used specifically in the proof) is that with a $(k+1)$-ary relation over a universe of cardinality n, we can encode n^{k+1} "bits of information" — a given $(k+1)$-tuple may or may not be in the relation; with a k-ary relation over a universe of cardinality roughly $n^{(k+1)/k}$, we can again encode roughly $(n^{(k+1)/k})^k = n^{k+1}$ bits of information.

Incidentally, in the case $k = 1$, it is also true (and we will show) that if Isom $(\{(\overline{g_1}(\mathfrak{A}) : \mathfrak{A} \in \mathscr{A}\})$ is in $\mathscr{F}_1(\mathscr{S})$, then $\mathscr{A}$ is in $\mathscr{F}_2(\mathscr{S})$. However, for every $\mathscr{S}$, the converse is false. For, we can essentially reduce to the case $\mathscr{S} = \emptyset$ by only considering cardinalities of structures. And, the set of even positive integers is in $\mathscr{F}_2$, whereas the set of squares of even positive integers $(g_1(n) = n^2)$ is not in $\mathscr{F}_1$, since as already mentioned, $\mathscr{F}_1 = \mathscr{F}_1(\emptyset)$ contains only finite and cofinite sets.

We will make use of the following concept. Assume that $\mathscr{T}$ is a finite similarity type, that U and V are unary predicate symbols with $U \notin \mathscr{T}$, $V \notin \mathscr{T}$, that σ is a first-order $\mathscr{T} \cup \{U, V\}$-sentence, and that $f\colon \mathsf{Z}^+ \to \mathsf{Z}^+$. Then we say that $\exists\mathscr{T}\sigma$ *defines* f (*with respect to* U *and* V) if for each finite $\{U, V\}$-structure $\mathfrak{A}$ with $\overline{\overline{U^{\mathfrak{A}}}} > 0$ and $\overline{\overline{V^{\mathfrak{A}}}} > 0$.

$$\mathfrak{A} \models \exists\mathscr{T}\sigma \quad \text{iff} \quad \overline{\overline{U^{\mathfrak{A}}}} = f(\overline{\overline{V^{\mathfrak{A}}}}).$$

We call f *binary-definable* if there is some $\mathscr{T}$ which contains only binary predicate symbols and some U, V, σ such that $\exists\mathscr{T}\sigma$ defines f with respect to U and V. Of course, $\mathscr{T}$ may be allowed to contain unary predicate symbols also, since a unary predicate symbol can be simulated by a binary predicate symbol.

We remark that our notion of definability is similar to that of TRAHTENBROT [8], but that there are very essential differences.

Lemma 8. *Assume that f_1 and f_2 are binary-definable. Then so are the sum $f_1 + f_2$ and the product $f_1 \cdot f_2$. If also $f_1(n) \geqq n$ for each n, then the composition $f_1 \circ f_2$ is binary-definable.*

Proof. Assume that $\exists\mathscr{T}_i\sigma_i(U_i, V_i)$ defines f_i with respect to U_i and V_i $(i = 1, 2)$; we can assume that $\mathscr{T}_1$ and $\mathscr{T}_2$ are disjoint, and that neither contains U, V, U_1, U_2, V_1, or V_2. Then

$\exists\mathscr{T}_1\,\exists\mathscr{T}_2\,\exists U_1\,\exists U_2(\sigma_1(U_1, V) \wedge \sigma_2(U_2, V) \wedge$ "U is the disjoint union of U_1 and U_2")

clearly defines $f_1 + f_2$ with respect to U and V.

Let P and Q be new binary predicate symbols. Then

$\exists\mathscr{T}_1\,\exists\mathscr{T}_2\,\exists P\,\exists Q\,\exists U_1\,\exists U_2(\sigma_1(U_1, V) \wedge \sigma_2(U_2, V) \wedge$ "there is a one-one correspondence between pairs $\langle u_1, u_2\rangle$ with u_1 in U_1, u_2 in U_2 and points u in U via $(Puu_1 \wedge Quu_2)$")

clearly defines $f_1 \cdot f_2$ with respect to U and V.

Finally, if $f_1(n) \geqq n$ for each n, then $\exists\mathscr{T}_1\,\exists\mathscr{T}_2\,\exists V_1(\sigma_2(V_1, V) \wedge \sigma_1(U, V_1))$ clearly defines $f_1 \circ f_2$ with respect to U and V.

Lemma 9. *Let p be a nonzero polynomial with nonnegative integral coefficients. Then p is binary-definable.*

Proof. Clearly, the nonzero constant functions are binary-definable, as is the identity function $n \mapsto n$. But, p can be obtained from these functions by repeated multiplication and addition. So by Lemma 8, p is binary-definable.

Lemma 10. *For each positive integer* k, *the function* $n \mapsto [n^{1/k}]$ *is binary-definable* ($[x]$ *is the greatest integer not exceeding* x).

Proof. Let p_1 be the polynomial $n \mapsto n^k$ of degree k, and let p_2 be the polynomial $n \mapsto ((n+1)^k - n^k)$ of degree $k-1$. Then by Lemma 9, p_1 and p_2 are binary-definable. Assume that $\exists\mathscr{T}_i\sigma_i(U_i, V_i)$ defines p_i with respect to U_i and V_i ($i = 1, 2$); we can assume that $\mathscr{T}_1$ and $\mathscr{T}_2$ are disjoint, and that neither contains U, V, U_1, U_2, V_1, or V_2. Let σ be $\sigma_1(U_1, U) \wedge \sigma_2(U_2, U) \wedge \forall x(Vx \leftrightarrow (U_1x \vee U_2x)) \wedge \exists x(U_1x \wedge U_2x)$. We will show that it is "almost" true that $\exists\mathscr{T}_1\exists\mathscr{T}_2\exists U_1\exists U_2\sigma$ defines $n \mapsto [n^{1/k}]$ with respect to U and V, and we will explain the meaning of the word "almost".

Assume that $\mathfrak{A}$ is a finite $\mathscr{T}_1 \cup \mathscr{T}_2 \cup \{U_1, U_2, V\}$-structure with $\mathfrak{A} \models \sigma$ and with $\overline{\overline{U^{\mathfrak{A}}}} > 0$ and $\overline{\overline{V^{\mathfrak{A}}}} > 0$. Let $u = \overline{\overline{U^{\mathfrak{A}}}}$, $v = \overline{\overline{V^{\mathfrak{A}}}}$, $u_1 = \overline{\overline{U_1^{\mathfrak{A}}}}$, and $v_1 = \overline{\overline{V_1^{\mathfrak{A}}}}$. We want to show that $u = [v^{1/k}]$, that is, that $u^k \leqq v$ and $(u+1)^k > v$. We know that $u_1 = u^k$, and that $u_2 = (u+1)^k - u^k$. Since $U_1^{\mathfrak{A}} \subseteq V^{\mathfrak{A}}$, it follows that $u_1 \leqq v$, and so $u^k \leqq v$, which we wanted to show. Now $(u+1)^k = u_1 + u_2$. Since $V^{\mathfrak{A}} = U_1^{\mathfrak{A}} \cup U_2^{\mathfrak{A}}$, and since $U_1^{\mathfrak{A}} \cap U_2^{\mathfrak{A}} \neq \emptyset$, we know that $u_1 + u_2 > v$. So $(u+1)^k > v$, as desired.

Now assume that $\mathfrak{A}$ is a finite $\{U, V\}$-structure with $u = [v^{1/k}]$, where $u = \overline{\overline{U^{\mathfrak{A}}}}$ and $v = \overline{\overline{V^{\mathfrak{A}}}}$. It is easy to see that we can find $\mathfrak{B}$ such that $\mathfrak{B} \upharpoonright \{U, V\} = \mathfrak{A}$ and $\mathfrak{B} \models \sigma$, as long as $p_2(u) \leqq v$. The polynomial p_2 is of degree $k-1$, and so if u is sufficiently large, then $p_2(u) < u^k \leqq v$, as desired. So for sufficiently large u, there is no problem; this is the meaning of "almost". We can take care of the finite number of cases when u is small by an obvious "finite modification" of σ.

We can now prove Theorem 7. Since the proof is long, we will split the theorem into two halves, Theorem 11 and Theorem 12.

Theorem 11. *Assume that* $k \geqq 2$, *that* g_k *is as in Theorem* 7, *and that* $\mathscr{A} \in \mathscr{F}_{k+1}(\mathscr{S})$. *Then* $\mathrm{Isom}(\{\overline{g_k}(\mathfrak{A}) : \mathfrak{A} \in \mathscr{A}\})$ *is in* $\mathscr{F}_k(\mathscr{S})$.

Proof. Let $\mathscr{B} = \mathrm{Isom}(\{\overline{g_k}(\mathfrak{A}) : \mathfrak{A} \in \mathscr{A}\})$. We want to show that $\mathscr{B} \in \mathscr{F}_k(\mathscr{S})$. The essential idea is as follows. Assume that $\mathscr{A} = Mod_\omega \exists\mathscr{T}\sigma$, where $\mathscr{T}$ is a set of t distinct k-ary predicate symbols $R_1, \ldots, R_t$ not in $\mathscr{S}$. Assume that $\mathfrak{B} \in \mathscr{B}$. We want to simulate $Rx_1 \ldots x_{k+1}$, for each R in $\mathscr{T}$, where each x_i runs through a subuniverse A of $|\mathfrak{B}|$ of cardinality n. We let variables v_i run through a yet smaller universe B of cardinality $[n^{1/k}]$, and let variables y_i run through the large universe U of $\mathfrak{B}$ of cardinality $n[n^{1/k}]$. We essentially set up (with a modification to be described soon) a one-one correspondence between points in A and k-tuples of points in B, and a one-one correspondence between pairs $\langle a, b\rangle$ where $a \in A$, $b \in B$, and points in U. For each $(k+1)$-ary predicate symbol R, we let $\bar{R}$ be a new k-ary predicate symbol. We simulate $Rx_1 \ldots x_{k+1}$ by $\bar{R}y_1 \ldots y_k$, where x_{k+1} corresponds (in the one-one correspondence) to $\langle v_1, \ldots, v_k\rangle$, and $\langle x_i, v_i\rangle$ corresponds to y_i ($1 \leqq i \leqq k$). A slight modification is called for, since $\bar{\bar{B}}^k$ may be less than $\bar{\bar{A}}$: we use two different correspondences between points in A and certain k-tuples of points in B, so that we can "double-use" some of the k-tuples; this takes care of the problem, since we will see that $2[n^{1/k}]^k \geqq n$ for sufficiently large n. Because of this modification, we will need to make slight technical adjustments. We now begin the formal construction.

Let U, A_1, A_2, A, and B be new unary predicate symbols; $S_1^i, \ldots, S_k^i$, and T_i new binary predicate symbols ($i = 1, 2$); G_1 and G_2 new k-ary predicate symbols; $\mathscr{T}'$ a certain finite set of new binary predicate symbols defined below; $\overline{\mathscr{T}_1}$ a set of new k-ary predicate symbols $\bar{R}_1^1, \ldots, \bar{R}_t^1$; and $\overline{\mathscr{T}_2}$ a set of t new k-ary predicate symbols

$\bar{R}^2_1, \ldots, \bar{R}^2_t$. Let $\mathscr{T}_0$ be the set of all these new symbols; we assume that they are distinct and differ from the symbols in $\mathscr{S}$ and $\mathscr{T}$.

Assume that $\exists\mathscr{T}'\tau_1$ defines $n \mapsto [n^{1/k}]$ with respect to B and A; this is possible by Lemma 10. Let τ_2 be the sentence

"There is a one-one correspondence between pairs $\langle a, b\rangle$ with a in A and b in B, and points u in U, via $(T_1ua \wedge T_2ub)$".

Then $\exists\mathscr{T}'\, \exists B\, \exists T_1\, \exists T_2(\tau_1 \wedge \tau_2)$ defines $n \mapsto n[n^{1/k}]$ with respect to U and A.

As before, for each P in $\mathscr{S}$, if P is r-ary, then let α_P be the sentence

$$\forall z_1 \ldots \forall z_r(Pz_1 \ldots z_r \rightarrow \bigwedge_{i=1}^{r} Az_i).$$

Let τ_3 be the sentence

"A is the disjoint union of A_1 and A_2, and there is a one-one correspondence between points x in A_i and k-tuples $\langle v_1, \ldots, v_k\rangle$ of points in B for which $G_iv_1 \ldots v_k$, via $\bigwedge_{m=1}^{k} S^i_mxv_m$ $(i = 1, 2)$".

Let ψ_1 be the following formula:

$$\bigvee_{i=1}^{2} (A_ix \wedge \bigwedge_{j=1}^{k} Bv_j \wedge G_iv_1 \ldots v_k \wedge \bigwedge_{j=1}^{k} S^i_jxv_j).$$

So, ψ_1 says that x in A corresponds to the k-tuple $\langle v_1, \ldots, v_k\rangle$ of points in B.

Let ψ_2 be the formula $Ax \wedge Bv \wedge T_1yx \wedge T_2yv$. So, ψ_2 says that the point y in U (the universe) corresponds to the pair $\langle x, v\rangle$, where $x \in A$ and $v \in B$.

We are now going to define a sentence σ', which simulates the statement that σ is true about a structure with universe A.

Let X be the set of all variables that occur in σ. Let $V = \{v^i_x\colon x \in X,\ 1 \leqq i \leqq k\}$ be a set of $k\bar{\bar{X}}$ new variables (x in A will correspond the the k-tuple $\langle v^1_x, \ldots, v^k_x\rangle$ of points in B). Let $Y = \{y_s\colon s \in X \times V\}$ be a set of $\bar{\bar{X}} \cdot \bar{\bar{V}}$ new variables ($y_{\langle x,v\rangle}$ in U will correspond to the pair $\langle x, v\rangle$, where $x \in A$, $v \in B$). Assume that all of these variables are distinct.

Let $\Phi = \{ \bigwedge_{x\in X} \gamma_x\colon \gamma_x$ is either A_1x or $A_2x\}$. Thus, Φ contains $2^{\bar{\bar{X}}}$ distinct formulas.

We can assume without loss of generality that σ is in prenex normal form $Q_1x_1 \ldots Q_mx_mM$, where each Q_i is $\forall$ or $\exists$ $(1 \leqq i \leqq m)$, where x_i and x_j are distinct variables if $i \neq j$, and where M is quantifier-free. For each φ in Φ, let M_φ be the result of the following substitutions into M. If $Rx_{j_1} \ldots x_{j_{k+1}}$ occurs in M (where $R \in \mathscr{T}$), and if φ contains $A_ix_{j_{k+1}}$ as a conjunct, then replace $Rx_{j_1} \ldots x_{j_{k+1}}$ in M by $\bar{R}^iy_{s_1} \ldots y_{s_k}$, where s_m is $\langle x_{j_m}, v^m_z\rangle$, for $1 \leqq m \leqq k$ and $1 \leqq i \leqq 2$, and where z is $x_{j_{k+1}}$. Exactly one of the two cases $i = 1$ or $i = 2$ takes place. Let M' be the following formula:

$$\forall \boldsymbol{v}\, \forall \boldsymbol{y}((\bigwedge_{x\in X} \psi_1(x, v^1_x, \ldots, v^k_x) \wedge \bigwedge_{x\in X,\, v\in V} \psi_2(y_{\langle x,v\rangle}, x, v)) \rightarrow \bigvee_{\varphi\in\Phi} \varphi \wedge M_\varphi),$$

where $\forall \boldsymbol{v}\, \forall \boldsymbol{y}$ means a universal quantification over every v in V and every y in Y.

Inductively define formulas σ'_i $(0 \leqq i \leqq m)$, by letting $\sigma_0 = M'$, and defining σ'_{i+1} as follows $(0 \leqq i < m)$:

$$\sigma'_{i+1} = \begin{cases} \forall x_{m-i}(Ax_{m-i} \rightarrow \sigma'_i), & \text{if } Q_{m-i} \text{ is } \forall \\ \exists x_{m-i}(Ax_{m-i} \wedge \sigma'_i), & \text{if } Q_{m-i} \text{ is } \exists. \end{cases}$$

Let $\sigma' = \sigma'_m$, and let $\mathscr{B}' = Mod_\omega\, \exists\mathscr{T}(\tau_1 \wedge \tau_2 \wedge \tau_3 \wedge \sigma' \wedge \forall x Ux \wedge \bigwedge_{P\in\mathscr{S}} \alpha_P)$.

It is straightforward to check that $\mathscr{B}$ is a finite modification of $\mathscr{B}'$; so, since $\mathscr{B}' \in \mathscr{F}_k(\mathscr{S})$, so is $\mathscr{B}$. The only difficulty lies in showing that $2[n^{1/k}]^k \geqq n$ for sufficiently large n, so that we can set up the correspondences we want. This is equivalent to showing that $2^{1/k}[n^{1/k}] \geqq n^{1/k}$ for sufficiently large n. Write $2^{1/k} = 1 + \theta$; so, $\theta > 0$. Let n be large enough to insure that $\theta[n^{1/k}] \geqq 1$. Then $2^{1/k}[n^{1/k}] \geqq [n^{1/k}] + 1 > n^{1/k}$, which was to be shown.

Theorem 12. *Assume that* $k \geqq 1$, *that* $\mathscr{A}$ *and* g_k *are as in Theorem* 7, *and that* $\mathrm{Isom}(\{\overline{g_k}(\mathfrak{A}) \colon \mathfrak{A} \in \mathscr{A}\})$ *is in* $\mathscr{F}_k(\mathscr{S})$. *Then* $\mathscr{A} \in \mathscr{F}_{k+1}(\mathscr{S})$.

Proof. We will prove the following result:

(1) Assume that $\mathscr{A}_1 \in \mathscr{F}_k(\mathscr{S})$, and that $\mathscr{B}_1 = \{\mathfrak{B} \in Fin(\mathscr{S}) \colon \overline{g_k}(\mathfrak{B}) \in \mathscr{A}_1\}$. Then $\mathscr{B}_1 \in \mathscr{F}_{k+1}(\mathscr{S})$.

Then Theorem 12 follows. For, let $\mathscr{A}_1 = \mathrm{Isom}(\{\overline{g_k}(\mathfrak{A}) \colon \mathfrak{A} \in \mathscr{A}\})$, and let $\mathscr{B}_1 = \{\mathfrak{B} \in Fin(\mathscr{S}) \colon \overline{g_k}(\mathfrak{B}) \in \mathscr{A}_1\}$. Then $\mathscr{B}_1 \in \mathscr{F}_{k+1}(\mathscr{S})$, by (1). Now g_k is strictly monotone, and hence one-one. So $\overline{g_k}$ is essentially one-one, by Lemma 2. It follows easily that $\mathscr{B}_1 = \mathscr{A}$.

We will now prove (1). Let $\mathscr{A}_1 = Mod_\omega\, \exists\mathscr{T}\sigma$, where $\mathscr{T}$ is a set of t distinct k-ary predicate symbols $R_1, \ldots, R_t$. Let $\mathscr{B}_1 = \{\mathfrak{B} \in Fin(\mathscr{S}) \colon \overline{g_k}(\mathfrak{B}) \in \mathscr{A}_1\}$. We want to show that $\mathscr{B}_1 \in \mathscr{F}_{k+1}(\mathscr{S})$. The essential idea is as follows. Assume that $\mathfrak{B} \in \mathscr{B}_1$. Let U be the universe of $\mathfrak{B}$, of cardinality n, let B be a set of cardinality $[n^{1/k}]$, and let A be a set of cardinality $[n^{1/k}]^k$. For each R in $\mathscr{T}$, we want to simulate $Rx_1 \ldots x_k$, where each x_i ranges through an imaginary larger universe of cardinality $n[n^{1/k}]$, whose "points" are pairs $\langle u, b\rangle$, with u in U and b in B. We set up a one-one correspondence between points in A and k-tuples of points in B. For each k-ary predicate symbol R, we let $\bar{R}$ be a new $(k + 1)$-ary predicate symbol. We simulate $R(\langle u_1, b_1\rangle, \ldots, \langle u_k, b_k\rangle)$ by $\bar{R}u_1 \ldots u_k a$, where a in A corresponds to $\langle b_1, \ldots, b_k\rangle$.

Let A, B, and U be new unary predicate symbols; $S_1, \ldots, S_k$ new binary predicate symbols; $\mathscr{T}'$ a certain finite set of new binary predicate symbols, defined below; and $\bar{\mathscr{T}}$ a set of t distinct $(k + 1)$-ary predicate symbols $\bar{R}_1, \ldots, \bar{R}_t$. Let $\mathscr{T}_0$ be the set of all these new symbols; we assume that they are distinct and differ from the symbols in $\mathscr{S}$ and $\mathscr{T}$.

Assume that $\exists\mathscr{T}'\tau_1$ defines $n \mapsto [n^{1/k}]$ with respect to B and U; this is possible, by Lemma 10. Let τ_2 be the sentence

"There is a one-one correspondence between points x in A and k-tuples $\langle y_1, \ldots, y_k\rangle$ of points in B, via $\bigwedge_{i=1}^{k} S_i x y_i$".

Then $\exists\mathscr{T}'\, \exists B\, \exists S_1 \ldots \exists S_k(\tau_1 \wedge \tau_2)$ defines $n \mapsto [n^{1/k}]^k$ with respect to A and U.

Let ψ be the formula $Av \wedge \bigwedge_{i=1}^{k} Bx_i \wedge \bigwedge_{i=1}^{k} S_i v x_i$. So ψ says that v in A corresponds to the k-tuple $\langle x_1, \ldots, x_k\rangle$ of points in B.

We will now define a sentence σ', which simulates the statement that σ is true about the structure with universe consisting of pairs $\langle u, b\rangle$, with u in U and b in B.

Let X be the set of all variables which occur in σ. For each x in X, let x^1 and x^2 be two new variables (the imaginary universe will consist of pairs $\langle x^1, x^2\rangle$, with x^1 in U and x^2 in B). For each k-tuple s in X^k, let v_s be a new variable (the point $v_{\langle x_1,\ldots,x_k\rangle}$ in A corresponds to the k-tuple $\langle x_1^2, \ldots, x_k^2\rangle$ of points in B). Let w be yet another new variable (w is thought of as a fixed point in B, and the point x in U is represented by the pair $\langle x, w\rangle$).

Let $\Phi = \{\bigwedge_{x\in X} \gamma_x\colon \gamma_x$ is either $x^2 = w$ or $x^2 \neq w\}$. Thus, Φ contains $2^{\overline{\overline{X}}}$ distinct formulas.

We can assume that σ is in prenex normal form $Q_1x_1 \ldots Q_mx_mM$, where each Q_i is $\forall$ or $\exists$, where x_i and x_j are distinct variables if $i \neq j$, and where M is quantifier-free. For each $\varphi \in \Phi$, let M_φ be the result of the following substitutions into M. If $x_i = x_j$ appears in M, then replace it by $x_i^1 = x_j^1 \wedge x^2 = x^2$. If $Px_{i_1} \ldots x_{i_r}$ appears in M (where $P \in \mathscr{S}$), and if $x_{i_s}^2 = w$ is a conjunct of φ for each s ($1 \leqq s \leqq r$), then substitute $Px_{i_1}^1 \ldots x_{i_r}^1$ in M for $Px_{i_1} \ldots x_{i_r}$. If $Px_{i_1} \ldots x_{i_r}$ appears in M (where $P \in \mathscr{S}$), and if $x_{i_s}^1 \neq w$ is a conjunct of φ for some s ($1 \leqq s \leqq r$), then substitute $x_{i_1} \neq x_{i_1}$ in M for $Px_{i_1} \ldots x_{i_r}$. If $Rx_{i_1} \ldots x_{i_k}$ appears in M (where $R \in \mathscr{T}$), then substitute $\bar{R}x_{i_1}^1 \ldots x_{i_k}^1 v_s$ in M for $Rx_{i_1} \ldots x_{i_k}$, where $s = \langle x_{i_1}, \ldots, x_{i_k}\rangle$). Let M' be the following formula:

$$\forall \boldsymbol{v}\Big(\Big(\bigwedge_{\langle z_1,\ldots,z_k\rangle \in X^k} \psi(v_{\langle z_1,\ldots,z_k\rangle}, z_1, \ldots, z_k)\Big) \to \bigvee_{\varphi\in\Phi} \varphi \wedge M_\varphi\Big),$$

where $\forall \boldsymbol{v}$ means a universal quantification over every v_s with s in X^k.

Inductively define formulas σ_i' ($0 \leqq i \leqq m$), by letting $\sigma_0' = M'$, and defining σ_{i+1}' as follows ($0 \leqq i < m$):

$$\sigma_{i+1}' = \begin{cases} \forall x_{m-i}^1\, \forall x_{m-i}^2((Ax_{m-i}^1 \wedge Bx_{m-i}^2) \to \sigma_i'), & \text{if } \ Q_{m-i} \text{ is } \forall \\ \exists x_{m-i}^1\, \exists x_{m-i}^2(Ax_{m-i}^1 \wedge Bx_{m-i}^2 \wedge \sigma_i'), & \text{if } \ Q_{m-i} \text{ is } \exists. \end{cases}$$

Let σ' be $(\exists w \in B)\, \sigma_m'$. It is straightforward to check that $\mathscr{B}_1 = Mod_\omega \exists \mathscr{T}_0(\tau_1 \wedge \wedge \tau_2 \wedge \sigma' \wedge \forall x Ux)$. Hence $\mathscr{B}_1 \in \mathscr{F}_{k+1}(\mathscr{S})$.

Clearly, Theorem 7 follows from Theorems 11 and 12.

In the special case when $\mathscr{S} = \emptyset$, we get the following result from Theorem 7.

Corollary 13. *Assume that* $k \geqq 2$ *and* $S \subseteqq \mathsf{Z}^+$. *Then* $S \in \mathscr{F}_{k+1}$ *iff* $\{n[n^{1/k}]\colon n \in S\}$ *is in* $\mathscr{F}_k$.

Proof. Immediate from Theorem 7.

We conclude this section with a consequence of Theorems 3 and 12. We need two preliminary lemmas. If $h\colon \mathsf{Z}^+ \to \mathsf{Z}^+$ is a function, then define $h^{(m)}$ inductively, by letting $h^{(1)} = h$ and letting $h^{(m+1)}$ be the composition $h \circ h^{(m)}$.

Lemma 14. *Let* g^p *be the function* $n \mapsto n[n^{1/p}]$. *For each pair* p, k *of positive integers, there are positive integers* m *and* N *such that* $g_p^{(m)}(n) \geqq n^k$ *for each* $n \geqq N$.

Proof. It is straightforward to show that $g^p(n) \geqq n^{1+1/(2p)}$ for sufficiently large n. The lemma now follows.

Lemma 15. *For each pair of positive integers* p, k, *there is a positive integer* m *such that whenever* $\mathscr{A} \in \mathscr{F}_k(\mathscr{S})$, *then* $\mathrm{Isom}(\{\overline{g_p^{(m)}}(\mathfrak{A})\colon \mathfrak{A} \in \mathscr{A}\})$ *is in* $\mathscr{F}_2(\mathscr{S})$.

Proof. Assume that $\mathscr{A} \in \mathscr{F}_k(\mathscr{S})$. Let f_k be the function $n \mapsto n^k$, and let $\mathscr{B} = \mathrm{Isom}(\{\overline{f_k}(\mathfrak{A})\colon \mathfrak{A} \in \mathscr{A}\})$. From Theorem 3, we know that $\mathscr{B} \in \mathrm{BIN}(\mathscr{S})$; let $\mathscr{B} = Mod_\omega \exists Q\sigma$, where Q is a new binary predicate symbol.

Find m from Lemma 14 such that $g_p^{(m)}(n) \geqq n^k$ for sufficiently large n. Let A, B, U be new unary predicate symbols, let $\mathscr{T}_1$ and $\mathscr{T}_2$ be disjoint sets of new binary predicate symbols, and let τ_1 and τ_2 be first-order sentences such that $\exists\mathscr{T}_1\tau_1$ defines $n \mapsto n^k$ with respect to B and A, and $\exists\mathscr{T}_2\tau_2$ defines $g_p^{(m)}$ with respect to U and A; this is possible, by Lemmas 8, 9, and 10. Let $\mathscr{T}_0$ be $\mathscr{T}_1 \cup \mathscr{T}_2 \cup \{A, B, U, Q\}$.

For each P in $\mathscr{S}$(P r-ary) let α_P be the sentence $\forall x_1 \ldots \forall x_r(Px_1 \ldots x_r \rightarrow \bigwedge_{i=1}^{r} Ax_i)$.

Let $\mathscr{C} = Mod_\omega\, \exists\mathscr{T}_0(\tau_1 \wedge \tau_2 \wedge \sigma^B \wedge \forall x Ux \wedge \bigwedge_{P\in\mathscr{S}} \alpha_p)$. Then $\mathscr{C} \in \mathscr{F}_2(\mathscr{S})$. So, we need only show that $\mathrm{Isom}(\{\overline{g_p^{(m)}}(\mathfrak{A})\colon \mathfrak{A} \in \mathscr{A}\})$ is a finite modification of $\mathscr{C}$. But this is true, because if $\mathfrak{C} \in \mathscr{C}$, and if card ($\mathfrak{C}$) is sufficiently large, then $\mathfrak{C}$ contains substructures $\mathfrak{A} \subseteqq \mathfrak{B}$, with $\overline{f_k}(\mathfrak{A}) = \mathfrak{B}$, $\overline{g_p^{(m)}}(\mathfrak{A}) = \mathfrak{C}$, and $\mathfrak{B}$ in $\mathscr{B}$ (and hence $\mathfrak{A}$ in $\mathscr{A}$).

Theorem 16. *Assume that* $\mathscr{F}_p(\mathscr{S}) = \mathscr{F}_{p+1}(\mathscr{S})$. *Then* $\mathscr{F}_k(\mathscr{S}) = \mathscr{F}_p(\mathscr{S})$ *for each* $k \geqq p$.

Proof. Case 1: $p = 1$. Then $\mathscr{F}_1(\mathscr{S}) \neq \mathscr{F}_2(\mathscr{S})$. For, let $\mathscr{A} = \{\mathfrak{A} \in Fin(\mathscr{S})\colon \mathrm{card}(\mathfrak{A})$ is even$\}$. Then as in the commentary following the statement of Theorem 7, we find that $\mathscr{A} \in \mathscr{F}_2(\mathscr{S})$, but $\mathscr{A} \notin \mathscr{F}_1(\mathscr{S})$.

Case 2: $p \geqq 2$. Assume that $\mathscr{F}_p(\mathscr{S}) = \mathscr{F}_{p+1}(\mathscr{S})$. Then we will prove that the following statement holds for each m:

(2) Assume that $\mathrm{Isom}(\{\overline{g_p^{(m)}}(\mathfrak{A})\colon \mathfrak{A} \in \mathscr{A}\})$ is in $\mathscr{F}_p(\mathscr{S})$. Then $\mathscr{A} \in \mathscr{F}_p(\mathscr{S})$.

Statement (2) holds for $m = 1$ by Theorem 12 (with $k = p$), since we are assuming that $\mathscr{F}_p(\mathscr{S}) = \mathscr{F}_{p+1}(\mathscr{S})$. Then (2) follows for general m by a straightforward induction.

Now assume that $\mathscr{A} \in \mathscr{F}_k(\mathscr{S})$ for some $k \geqq p$. Find m from Lemma 15 such that $\mathrm{Isom}(\{\overline{g_p^{(m)}}(\mathfrak{A})\colon \mathfrak{A} \in \mathscr{A}\})$ is in $\mathscr{F}_2(\mathscr{S}) \subseteqq \mathscr{F}_p(\mathscr{S})$. Then from (2), we find that $\mathscr{A} \in \mathscr{F}_p(\mathscr{S})$. So $\mathscr{F}_k(\mathscr{S}) \subseteqq \mathscr{F}_p(\mathscr{S})$. But also $\mathscr{F}_p(\mathscr{S}) \subseteqq \mathscr{F}_k(\mathscr{S})$, since $p \leqq k$. Hence $\mathscr{F}_k(\mathscr{S}) = \mathscr{F}_p(\mathscr{S})$.

Conjecture. If $p \neq k$, then $\mathscr{F}_p(\mathscr{S}) \neq \mathscr{F}_k(\mathscr{S})$.

Bibliography

[1] Asser, G., Das Repräsentantenproblem im Prädikatenkalkül der ersten Stufe mit Identität. This Zeitschr. **1** (1955), 252–263.

[2] Fagin, R., Contributions to the model theory of finite structures. Doctoral dissertation, Univ. of Calif., Berkeley 1973.

[3] Fagin, R., Generalized first-order spectra and polynomial-time recognizable sets. In: Complexity of Computation (ed. R. Karp), SIAM-AMS proc. **7** (1974), 43–73.

[4] Fagin, R., Monadic generalized Spectra. This Zeitschr. **21** (1975), 89–96.

[5] Rabin, M. O., A simple method for undecidability proofs and some applications. Logic, methodology, and philosophy of science. Proc. of the 1963 International Congress, ed Y. Bar-Hillel, North-Holland, pp. 58–68.

[6] Scholz, H., Problems. J. Symb. Log. **17** (1952), 160.

[7] Tarski, A., Contributions to the theory of models I, II. Indag. Math. **16** (1954), 572–588.

[8] Trahtenbrot, B. A., Névozmožnost algoritma dlá problémy razrěšimosti na konéčnyh klassah. (Impossibility of an algorithm for the decision problem in finite classes). Doklady Akadémii Nauk SSSR **70** (1950), 569–572.

(Eingegangen am 6. Dezember 1973)